K-THEORY OF C^*-ALGEBRAS

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K-theory of C^*-algebras appeared in the 1970ies as a noncommutative counterpart of Atiyah-Hirzebruch topological K-theory. In some sense, this theory may be viewed as “algebraic topology for C^*-algebras”. K-theory naturally associates two abelian groups, $K_0(A)$ and $K_1(A)$, to every C^*-algebra A. These groups are quite important invariants of A. On the one hand, they contain much information about A, and on the other hand, there are powerful tools to explicitly calculate them. If $A = C(X)$, the algebra of continuous functions on a compact Hausdorff topological space X, then $K_0(A)$ and $K_1(A)$ are just the topological K-groups $K^0(X)$ and $K^1(X)$, respectively. Thus topological K-theory is fully embedded into K-theory of C^*-algebras. A number of fundamental results in topological K-theory, including the Bott periodicity, have natural extensions to C^*-algebras. At the same time, K-theory of C^*-algebras has some interesting “purely noncommutative” properties, which do not have classical prototypes. In this course we define K-theory for C^*-algebras, prove its basic properties (including the Bott periodicity theorem), and calculate the K-groups in some important cases.

Prerequisites. The basics of functional analysis (Banach and Hilbert spaces, bounded linear operators). Some acquaintance with C^*-algebras will be helpful (for example, as given in the first half of the course “C^*-algebras and compact quantum groups”, spring 2024, or in the respective part of the course “Harmonic analysis and Banach algebras”, fall 2024). Anyway, basic facts on C^*-algebras will be surveyed in the beginning of the course.

Syllabus

1. Basic facts on C^*-algebras (a survey).
2. Equivalence relations for projections. The group $K_0(A)$. Remarks on the commutative case (vector bundles, the Serre-Swan theorem, topological K-theory).
3. Homotopy invariance, half-exactness, and stability of K_0.
4. Equivalence of unitaries. The group $K_1(A)$.
5. The index map in K-theory. A relation to the Fredholm index. The exact sequence of K-groups induced by a C^*-algebra extension.
6. The Toeplitz algebra. The Bott periodicity.
7. Inductive limits of C^*-algebras. The continuity of K_0. The order structure on $K_0(A)$. AF-algebras and their Bratteli diagrams. Elliott's classification of AF-algebras in terms of K-theory.