$G =$ LCA group (2nd countable).

$\widehat{G} = \mathrm{Hom}_{\text{cont}}(G, \mathbb{T})$.

$f, g \in C_b(G)$

$(f \otimes g)(x,y) = f(x)g(y)$

$C_b(G) \otimes C_b(G) \hookrightarrow C_b(G \times G)$

$\widehat{G} \subset C(G)$ Pontryagin top. on $\widehat{G}$.

$\nu \in M(G)$ $\quad \hat{\nu} : \widehat{G} \to \mathbb{C}$ $\quad \hat{\nu}(\chi) = \int_G \chi \, d\nu = \langle \nu, \chi \rangle$

$\hat{\nu} \in C_b(\widehat{G})$, $\|\hat{\nu}\|_\infty \leq \|\nu\|$.

$\|\nu\| = |\nu|(G)$

norm on $M(G)$

$\mathcal{F}_G = \mathcal{F} : M(G) \to C_b(\widehat{G})$, $\nu \mapsto \hat{\nu}$

bdd lin map

Fourier transform.

Example. δ_x = Dir. meas concentrated at $x \in G$.

$\hat{\delta}_x(\chi) = \chi(x)$, that is, $\hat{\delta}_x = \varepsilon_x$ (eval. at x)

In part, $\hat{\delta}_e = 1$

$(Sf)(x) = f(x^{-1})$

Prop. $\mathcal{F} : M(G) \to C_b(\hat{G})$ is a unital $*$-alg. hom.

Proof $(\nu_1 * \nu_2)^\wedge(\chi) = \langle \nu_1 * \nu_2, \chi \rangle = \langle \nu_1 \otimes \nu_2, \Delta\chi \rangle$

$= \langle \nu_1 \otimes \nu_2, \chi \otimes \chi \rangle = \langle \nu_1, \chi \rangle \langle \nu_2, \chi \rangle = \hat{\nu}_1(\chi)\hat{\nu}_2(\chi)$

$(\nu^*)^\wedge(\chi) = \langle \nu^*, \chi \rangle = \overline{\langle \nu, \overline{S\chi} \rangle} = \overline{\langle \nu, \chi \rangle} = \overline{\hat{\nu}(\chi)}$. $\square$

μ = Haar meas on G

$L^1(G) = L^1(G,\mu) \hookrightarrow M(G)$, $f \mapsto f \cdot \mu$.

<u>Def</u> The <u>Fourier transform</u> of $f \in L^1(G)$ is $\hat{f} = (f \cdot \mu)^\wedge$.

$$\hat{f}(\chi) = \int_G f \chi \, d\mu.$$

<u>Construction.</u> $\chi \in \hat{G}$.

Define $\tilde{\chi} : M(G) \to \mathbb{C}$, $\tilde{\chi}(\nu) = \hat{\nu}(\chi)$.

$\tilde{\chi}$ is a unital $*$-character of $M(G)$

<u>Observe</u>: $\tilde{\chi}|_{L^1(G)} \neq 0$ (because $L^\infty \xrightarrow[\sim]{} (L^1)^*$).

<u>Notation</u>. $\gamma : \hat{G} \to L^1(G)^\wedge$, $\chi \mapsto \tilde{\chi}|_{L^1(G)}$.

<u>Thm 1</u>. γ is bijective.

<u>Lemma 1</u>. G = lc group, $f \in L^1(G)$
The map $G \to L^1(G)$, $x \in G \mapsto L_x f$, is cont.

<u>Proof</u>. True if $f \in C_c(G)$ (exer) (Hint: we "almost" proved this before).

Let $f \in L^1(G)$, $\varepsilon > 0$

Choose $g \in C_c(G)$ s.t. $\|f-g\|_1 < \varepsilon$

$\forall x \in G$ $\exists$ a nbhd $U \ni x$ s.t. $y \in U$ $\|L_x g - L_y g\| < \varepsilon$.

$$\Rightarrow \|L_x f - L_y f\|_1 \le \underbrace{\|L_x(f-g)\|_1}_{<\varepsilon} + \underbrace{\|L_x g - L_y g\|_1}_{<\varepsilon} + \underbrace{\|L_y(g-f)\|_1}_{<\varepsilon} < 3\varepsilon. \quad \square$$

<u>Lemma 2.</u> A = comm Ban. alg, $I \subset A$ closed ideal.

$\hat{A}_I = \{\chi \in \hat{A} : \chi|_I \neq 0\}$. Then $\hat{A}_I$ is an open subset of $\hat{A}$,

and $\hat{A}_I \xrightarrow{\alpha} \hat{I}$, $\chi \mapsto \chi|_I$, is a homeomorphism

Proof. Let $\chi \in \hat{A}_I$. Then $\forall\, b \in I$ $\quad \chi(a) = \dfrac{\varphi(ab)}{\varphi(b)} \quad (*)$
$\chi|_I = \varphi$ $\quad$ whenever $\varphi(b) \neq 0$.

$(*) \Rightarrow \alpha$ is inj.

Let $\varphi \in \hat{I}$. Choose $b \in I$ s.t. $\varphi(b) = 1$.
Define $\chi : A \to \mathbb{C}$ by $\chi(a) = \varphi(ab)$.
$\chi(a_1)\chi(a_2) = \varphi(a_1 b)\varphi(a_2 b) = \varphi(a_1 b a_2 b) = \varphi(a_1 a_2 b)\varphi(b) = \chi(a_1 a_2)$.
$\Rightarrow \chi$ is a char of A, and $\chi|_I = \varphi$. $\Rightarrow \alpha$ is bijective.

Exer. Show that $\hat{A}_I$ is open, α and α^{-1} are cont. $\square$
(use $(*)$)

G = LCA group, $A = M(G)$, $I = L^1(G)$.

$$\hat{G} \xrightarrow{\beta} \hat{A}_I \xrightarrow{\alpha} \hat{I} \qquad \beta(\chi) = \tilde{\chi}. \qquad \alpha\beta = \gamma.$$

$$\alpha(\psi) = \psi|_I. \qquad \boxed{\tilde{\chi}(\nu) = \hat{\nu}(\chi)}$$

<u>Lemma 3</u> β is bijective.

<u>Proof</u>. $\forall \chi \in \hat{G}$ $\quad \chi(x) = \tilde{\chi}(\delta_x)$.

$\Longrightarrow \beta$ is injective.

Take $\varphi \in \hat{A}_I$. Define $\chi : G \to \mathbb{C}$, $\chi(x) = \varphi(\delta_x)$.

$$\chi(xy) = \varphi(\delta_{xy}) = \varphi(\delta_x * \delta_y) = \chi(x)\chi(y) \ ; \ \chi(e) = 1$$

$\chi(x)\chi(x^{-1}) = \chi(e) = 1 \Rightarrow \chi(x) \neq 0 \ \forall x. \Longrightarrow$

$\chi : G \to \mathbb{C}^\times$ is a char.

$|\chi(x)| \leq \|\delta_x\| = 1 \ \forall x \Longrightarrow \left|\frac{1}{\chi(x)}\right| = |\chi(x^{-1})| \leq 1 \Longrightarrow |\chi(x)| = 1.$

Choose $h \in L^1(G)$ st. $\varphi(h) = 1$.

$\Longrightarrow \chi(x) = \varphi(\delta_x)\varphi(h) = \varphi(\delta_x * h) = \varphi(L_x h) \overset{L1}{\Longrightarrow} \chi$ is cont

$\Longrightarrow \chi \in \hat{G}$.

We want: $\tilde{\chi} = \varphi$. L2. $\Rightarrow$ it suff to show that $\tilde{\chi}(f) = \varphi(f) \ \forall f \in L^1(G)$

$\exists g \in L^\infty(G)$ s.t. $\varphi(f) = \int_G f g \, d\mu \quad \forall f \in L^1(G)$.

$$\varphi(f) = \varphi(f)\varphi(h) = \varphi(f * h) = \int (f*h) g \, d\mu =$$

$$= \int \int f(y) h(y^{-1}x) g(x) \, d\mu(y) \, d\mu(x) = \int f(y) \underbrace{\left(\int (L_y h)(x) g(x) \, d\mu(x) \right)}_{\parallel} d\mu(y)$$

$$\varphi(L_y h) = \varphi(\delta_y * h) = \\ = \varphi(\delta_y)\varphi(h) = \chi(y)$$

$$= \int f(y) \chi(y) \, d\mu(y) = \tilde{\chi}(f).$$

$\Longrightarrow \beta$ is bijective. $\square$

Thm 1 follows from L2 & L3.

Cor. $L^1(G)$ is hermitian.

Proof $\forall \chi \in \hat{G}$ $\tilde{\chi}$ is a $*$-char of $L^1(G)$

$\Rightarrow$ all chars of $L^1(G)$ are $*$-chars $\Rightarrow L^1(G)$ is herm. $\square$.

Thm 2. $\gamma : \hat{G} \longrightarrow L^1(G)^\wedge$ is a homeomorphism.

Lemma 1 γ is cont.

Proof It suff to show that $\chi \longmapsto \underset{\hat{f}(\chi)}{\underset{\shortparallel}{\tilde{\chi}(f)}}$ is cont. $\forall f \in L^1$.

This is true $\square$

Lemma 2. E = normed space, $B \subset E^*$ bounded. Then

$$(B, wk^*) \times E \to \mathbb{C}, \quad (f, x) \mapsto f(x), \text{ is cont.}$$

Proof Let $C = \sup_{f \in B} \|f\|$. Let $f, f_0 \in B$, $x, x_0 \in E$.

$$|f(x) - f_0(x_0)| \leq |f(x - x_0)| + |f(x_0) - f_0(x_0)| \leq C\|x - x_0\| + \|f - f_0\|_{x_0} \quad \square$$

Notation $\hat{G}_w = (\hat{G}$; Gelf. top. induced from $L^1(G)^\wedge)$

Lemma 3 $\hat{G}_w \times G \to \mathbb{T}$, $(\chi, x) \mapsto \chi(x)$, is cont.

Proof. $\forall \chi \in \hat{G}$, $f \in L^1(G)$, $x \in G$.

$$\tilde{\chi}(L_x f) = \tilde{\chi}(\delta_x * f) = \chi(x)\tilde{\chi}(f). \Rightarrow \chi(x) = \frac{\tilde{\chi}(L_x f)}{\tilde{\chi}(f)}$$

if $\tilde{\chi}(f) \neq 0$.

Let $\chi_0 \in \hat{G}$.

Choose $f \in L^1(G)$ s.t. $\tilde{\chi}_0(f) \neq 0$.

$\exists$ a nbhd $U \ni \chi_0$ in $\hat{G}_w$ s.t. $\forall \chi \in U$ $\tilde{\chi}(f) \neq 0$.

By (*), it suff to show that

$$U \times G \to \mathbb{C}, \quad (\chi, x) \mapsto \tilde{\chi}(L_x f), \text{ is cont.}$$

$$U \times G \xrightarrow{\text{cont}} L^1(G)^\wedge \times L^1(G) \xrightarrow[\text{cont (L2)}]{\langle \cdot, \cdot \rangle} \mathbb{C} \qquad \square$$

$$(\chi, x) \longmapsto (\tilde{\chi}, L_x f) \qquad (\varphi, f) \mapsto \varphi(f)$$

<u>Lemma 4.</u> X, Y, Z top spaces, $F : X \times Y \to Z$ cont. $Z_0 \subset Z$ open, $Y_0 \subset Y$ compact. Then

$$\{x \in X : F(x,y) \in Z_0 \ \forall y \in Y_0\} \text{ is open in } X$$

<u>Lemma 5</u>: γ is open.

<u>Proof.</u> $\chi \in \hat{G}$, $K \subset G$ comp, $\varepsilon > 0$

$$U_{K,\varepsilon}(\chi) = \{\varphi \in \hat{G} : |\varphi(x) - \chi(x)| < \varepsilon \ \forall x \in K\}$$

(a basic open nbhd of χ)

We want: $U_{K,\varepsilon}(\chi)$ is open in $\hat{G}_w$.

L3: $(\varphi, x) \longmapsto |\varphi(x) - \chi(x)|$ is cont on $\hat{G}_w \times G$.

L2 $\Rightarrow$ $U_{K,\varepsilon}$ is open in $\hat{G}_w$. $\square$

Thm 2 follows from L1 & L5.

<u>Cor.</u> $\hat{G}$ is locally compact.

<u>Thm 3.</u> $\mathcal{F}(L^1(G)) \subset C_0(\hat{G})$, and the foll. diag commutes:

$$\begin{tikzcd}
 & C_0(\hat{G}) \\
L^1(G) \arrow[ur, "\mathcal{F}"] \arrow[dr, "\Gamma"'] & \\
 & C_0(L^1(G)^\wedge) \arrow[uu, "\wr"', "\gamma^*"]
\end{tikzcd}$$

Proof. $\forall f \in L^1(G)$ $\quad x \in \hat{G}$

$$(\gamma^*(\Gamma f))(x) = (\Gamma f \circ \gamma)(x) = (\Gamma f)(\gamma(x)) = (\Gamma f)(\tilde{x}) =$$

$$= \tilde{x}(f) = (\mathcal{F} f)(x). \quad \square$$

Cor. (density thm)

$\mathcal{F}(L^1(G))$ is dense in $C_0(\hat{G})$

Proof $L^1(G)$ is hermitian $\square$.