# Spectral properties of $C^*$-algebras

$A = *$-alg, $a \in A$

(hermitian)

**Def.** $a \in A$ is <u>selfadjoint</u> $\Longleftrightarrow a^* = a$.

$a$ is <u>normal</u> $\Longleftrightarrow aa^* = aa^*$.

If $A$ is unital, then $u \in A$ is <u>unitary</u> $\Longleftrightarrow u \in A^\times$ and $u^{-1} = u^*$.

<u>Observe</u>: (1) selfadj $\Longrightarrow$ normal

unitary $\Longrightarrow$

(2) $\forall a \in A$

$a^* a$ is selfadj.

<u>Notation.</u> $A_{sa} = \{a \in A : a = a^*\}$.

<u>Example 1</u> $A = \mathbb{C}^X$ or $A = \ell^\infty(X)$ or $A = C_b(X)$

(1) $f \in A$ is selfadj $\iff f(x) \in \mathbb{R}\ \forall x \in X$.

(2) $f \in A$ is unitary $\iff |f(x)| = 1\ \forall x \in X$.

<u>Example/exer 2.</u> $A = \mathcal{B}(H)$ ($H$ = Hilb sp)

(1) $T \in \mathcal{B}(H)$ is selfadj $\iff \langle Tx | x \rangle \in \mathbb{R}\ \forall x \in H$

(2) $U \in \mathcal{B}(H)$ is unitary $\iff U$ is bijective and

$\langle Ux | Uy \rangle = \langle x | y \rangle \quad (x, y \in H)$

**Prop.** $\forall a \in A$ $\exists$ a unique pair $(b,c)$ of selfadj. s.t. $a = b + ic$.

**Proof** $b = \dfrac{a+a^*}{2}$, $c = \dfrac{a-a^*}{2i}$. $\not\Leftarrow$ $\begin{cases} a = b + ic \\ a^* = b - ic \end{cases}$ $\square$.

**Thm 1** $A = C^*$-alg, $a \in A$ normal $\Longrightarrow r(a) = \|a\|$

**Proof** If $b \in A_{sa}$, then $\|b^2\| = \|b\|^2$.

Suppose $a \in A$ is normal.

$$\|a^*a\|^2 = \|(a^*a)^2\| = \|a^*aa^*a\| = \|(a^*)^2a^2\| = \|(a^2)^*a^2\| = \|a^2\|^2.$$

$$\overset{\shortparallel}{\|a\|^4} \qquad \Longrightarrow \|a\|^2 = \|a^2\|.$$

Induction $\Rightarrow \|a^{2^n}\| = \|a\|^{2^n}$.

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \|a\|. \qquad \square$$

<u>Cor. 1</u> $A = C^*$-alg $\Rightarrow \forall\, a \in A \quad \|a\| = \sqrt{r(a^*a)}$.

<u>Cor. 2.</u> If $A$ is a $*$-alg, then $\exists$ at most one norm on $A$ making $A$ into a $C^*$-alg.

Equivalently, every $*$-isomorphism between $C^*$-algebras is isometric.

**Cor. 3** $A =$ Ban. $*$-alg, $B = C^*$-alg. Then every $*$-hom $\varphi : A \to B$ is continuous, and $\|\varphi\| \leq 1$.

**Proof.** $\forall a \in A_{sa}$ $\varphi(a) \in B_{sa} \Longrightarrow$

$$\Longrightarrow \|\varphi(a)\| = r(\varphi(a)) \leq r(a) \leq \|a\|.$$

$\forall a \in A$

$$\|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| = \|\varphi(\underbrace{a^*a}_{A_{sa}})\| \leq \|a^*a\| \leq \|a\|^2. \quad \square$$

**Thm 2.** $A = C^*$-alg, $a \in A_{sa} \Longrightarrow \sigma'_A(a) \subset \mathbb{R}$

**Proof** We may assume that $A$ is unital.

Let $\lambda \in \sigma(a)$, $\lambda = \alpha + i\beta$ $(\alpha, \beta \in \mathbb{R})$.

$\forall t \in \mathbb{R}$ $\lambda + it \in \sigma(a + it1) \Longrightarrow$

$$|\lambda + it|^2 \leq \|a + it1\|^2 = \|(a - it1)(a + it1)\| = \|a^2 + t^2 1\| \leq \|a^2\| + t^2$$

$$|\lambda + it|^2 = \alpha^2 + (t+\beta)^2 = \alpha^2 + t^2 + 2t\beta + \beta^2$$

$$\Longrightarrow \alpha^2 + \beta^2 + 2\beta t \leq \|a^2\| \quad \forall t \in \mathbb{R}$$

$$\Longrightarrow \beta = 0. \quad \square$$

**Def** A $*$-alg $A$ is <u>hermitian</u> if $\forall a \in A_{sa}$ $\sigma'_A(a) \subset \mathbb{R}$.

**Examples.** (1) All $C^*$-alg.

(2) Every spec. inv $*$-subalg of a $C^*$-alg

For ex, $C^n[a,b]$ is herm.

**Exer.** Is $\mathcal{A}(\overline{\mathbb{D}})$ hermitian?

**Prop.** $A$ = herm. $*$-alg $\Longrightarrow$ all characters of $A$ are $*$-characters

<u>Proof</u> $\forall a \in A_{sa}$ $\sigma'_A(a) \subset \mathbb{R}$

$\chi : A \to \mathbb{C}$ char $\Longrightarrow \sigma'_{\mathbb{C}}(\chi(a)) \subset \mathbb{R}$, that is, $\chi(a) \in \mathbb{R}$.

$\forall a \in A$ $a = b + ic$ $(b, c \in A_{sa})$

$$\chi(a^*) = \chi(b - ic) = \underbrace{\chi(b)}_{\text{real}} - i\underbrace{\chi(c)}_{\text{real}} = \overline{\chi(b) + i\chi(c)} = \overline{\chi(a)}. \quad \square$$

**Thm 3.** $A$ = comm. Ban. $*$-alg. TFAE:

(1) $A$ is hermitian

(2) All characters of $A$ are $*$-characters

(3) $\Gamma_A : A \to C_0(\operatorname{Max} A)$ is a $*$-hom

Moreover, if $A$ is herm, then $\operatorname{Im}\Gamma_A$ is dense in $C_0(\operatorname{Max} A)$.

**Proof** $(1) \Rightarrow (2)$ see Prop.

(2) $\Longrightarrow$ (3)

$\forall a \in A \quad \forall \chi \in \hat{A} \qquad \widehat{a^*}(\chi) = \chi(a^*) = \overline{\chi(a)} = \overline{\hat{a}(\chi)} =$

$= \hat{a}^*(\chi)$

(3) $\Longrightarrow$ (1) $\forall a \in A_{sa}$

$\sigma'_A(a) = \hat{a}(\mathrm{Max}A) \cup \{0\} \subset \mathbb{R}.$

(all chars of A)

Let $B = \mathrm{Im}\,\Gamma_A \subset C_0(\mathrm{Max}A).$

$B_+ \subset C_0(\mathrm{Max}A)_+ \cong C((\mathrm{Max}A)_+) \cong C(\hat{A}_+)$

satisfies the conditions of the Stone-Weierstrass thm.

$\Rightarrow B_+$ is dense in $C(\hat{A}_+)$ $\underset{(\text{exer})}{\Longrightarrow}$ $B$ is dense in $C_0(\operatorname{Max} A)$. $\square$

<u>Thm</u> (Gelfand, Naimark)
$A$ = comm. $C^*$-alg $\Rightarrow \Gamma_A : A \to C_0(\operatorname{Max} A)$ is an isometric $*$-isomorphism.

<u>Proof</u>. We know: $\Gamma$ is a $*$-hom, $\operatorname{Im}\Gamma$ is dense in $C_0(\operatorname{Max} A)$

We have to show that $\Gamma$ is isometric.

$\forall a \in A_{sa}$

$$\|\Gamma(a)\| = r(a) = \|a\|.$$

$\forall a \in A$

$$\|\Gamma(a)\|^2 = \|\Gamma(a)^* \Gamma(a)\| = \|\Gamma(a^*a)\| = \|a^*a\| = \|a\|^2. \quad \square$$

## A category-theoretic interpretation.

$\mathcal{A}$, $\mathcal{B}$ categories $\qquad F : \mathcal{A} \to \mathcal{B}$ covariant functor.

**Def** $F$ is an <u>equivalence</u> if $\exists$ a cov. functor $G : \mathcal{B} \to \mathcal{A}$ s.t. $G \circ F \cong \mathbb{1}_{\mathcal{A}}$, $F \circ G \cong \mathbb{1}_{\mathcal{B}}$.

($G$ is a <u>quasi-inverse</u> of $F$)

**Notation** $CUC^*$ = the cat. of comm. unital $C^*$-alg.

Morphisms in $CUC^*$ = unital $*$-homoms.

**Thm.** $\mathrm{Comp}^{op} \underset{\mathrm{Max}}{\overset{C}{\rightleftarrows}} CUC^*$ are equivalences.

Moreover,

$$\mathrm{Max}\circ C \underset{\varepsilon}{\cong} \mathbb{1}_{\mathrm{Comp}^{op}}$$

$$C\circ \mathrm{Max} \underset{\Gamma}{\cong} \mathbb{1}_{\mathrm{CUC}^*}$$

$$\varepsilon_X : X \xrightarrow{\sim} \mathrm{Max}\, C(X)$$

# The Fourier trasform on loc. compact abelian groups

$G$ = loc comp abelian group. (2nd countable).

$\hat{G} = \mathrm{Hom}_{\mathrm{cont}}(G, \mathbb{T})$

$\hat{G}$ is an abelian group under the pointwise mult.

<u>Def.</u> $\hat{G}$ is the dual of $G$.

**Def.** The <u>Pontryagin topology</u> on $\widehat{G}$ is the restriction to $\widehat{G}$ of the compact-open topology on $C(G)$

Explicitly:

$\chi \in \widehat{G}$, $K \subset G$ compact, $\varepsilon > 0$.

$$U_{K,\varepsilon}(\chi) = \{ \varphi \in \widehat{G} : \| \varphi - \chi \|_K < \varepsilon \}$$

(where $\|f\|_K = \sup_{x \in K} |f(x)|$, $f \in C(G)$).

$\{U_{K,\varepsilon}(x) \mid K \subset G \text{ comp}, \varepsilon > 0\}$ is a base of open nbhds of $x \in \hat{G}$.

$$\left( U_{K_1,\varepsilon_1}(x) \cap U_{K_2,\varepsilon}(x) \supset U_{K,\varepsilon}(x) \quad \text{where } K = K_1 \cup K_2,\ \varepsilon = \min\{\varepsilon_1, \varepsilon_2\} \right).$$

hence this family is a base (and not only a subbase) of nbhds of $x$.

<u>Prop.</u> $\hat{G}$ is a top. group.

<u>Proof</u> (sketch). $\chi_1, \chi_2 \in \hat{G}$.

$$U_{K,\varepsilon}(\chi_1)\, U_{K,\varepsilon}(\chi_2) \subset U_{K,2\varepsilon}(\chi_1\chi_2) \quad \text{(exer.)}$$

$\Rightarrow$ the mult. on $\hat{G}$ is cont.

$$U_{K,\varepsilon}(\chi)^{-1} = U_{K,\varepsilon}(\chi^{-1}) \quad \text{(exer)} \quad (\chi^{-1} = \overline{\chi})$$

$\Rightarrow$ $\chi \mapsto \chi^{-1}$ is cont $\square$

**Def.** The **Fourier transform** of $\nu \in M(G)$ is

$$\hat{\nu} : \hat{G} \to \mathbb{C}, \quad \hat{\nu}(\chi) = \int_G \chi \, d\nu = \langle \nu, \chi \rangle.$$

$\mu$ = Haar meas on $G$

$$L^1(G) = L^1(G,\mu) \hookrightarrow M(G), \quad f \mapsto f \cdot \mu.$$

**Def** The **Fourier transform** of $f \in L^1(G)$ is

is $\hat{f} = \widehat{(f \cdot \mu)}$.

Explicitly

$$\hat{f}(\chi) = \int_G f \chi \, d\mu \,.\,.$$

<u>Observe</u>: $|\hat{\nu}(\chi)| \leq \int_G \underbrace{|\chi|}_{=1} \, d|\nu| = \|\nu\|.$

$\Rightarrow \hat{\nu}$ is bdd, and $\|\hat{\nu}\|_\infty \leq \|\nu\|.$

<u>Prop.</u> $\hat{\nu} \in C_b(\hat{G}).$

**Proof** Let $\chi_0 \in \widehat{G}$, $\varepsilon > 0$.

$\exists$ a compact set $K \subset G$ s.t. $|\nu|(G \setminus K) < \varepsilon$.

$\forall \chi \in U_{K,\varepsilon}(\chi_0)$

$$|\hat{\nu}(\chi) - \hat{\nu}(\chi_0)| \leq \int_G |\chi - \chi_0| \, d|\nu| = \int_K (\ldots) + \int_{G\setminus K} (\ldots) \leq$$

$$\leq \varepsilon \|\nu\| + 2\varepsilon = (\|\nu\| + 2)\varepsilon. \Longrightarrow \hat{\nu} \text{ is cont.} \quad \square$$

**Notation.** $\mathcal{F}_G : M(G) \to C_b(\widehat{G})$, $\nu \longmapsto \hat{\nu}$.

**Def.** $\mathcal{F}$ is the Fourier transform on $G$.

**Observe:** $\mathcal{F}$ is a bdd linear map