Unitization.

A = algebra.

$A_+ = A \oplus \mathbb{C}1_+$ (a vec space dir. sum)

Multiplication on A_+:

$$(a+\lambda 1_+)(b+\mu 1_+) = ab + \lambda b + \mu a + \lambda\mu 1_+.$$

A_+ becomes a unital alg.

<u>Def</u> A_+ is the <u>unitization</u> of A

Prop/exer 1. $A =$ alg, $B =$ unital alg; $\varphi : A \to B$ alg. hom.

(1) Define $\varphi_+ : A_+ \to B$ by $\varphi_+(a + \lambda 1_+) = \varphi(a) + \lambda 1_B$

Then φ_+ is a unital alg. hom.

(2) $\exists$ a natural bijection

$$\mathrm{Hom}_{\mathrm{Alg}}(A, B) \rightleftarrows \mathrm{Hom}_{\mathrm{Un.Alg}}(A_+, B)$$

$$\varphi \longmapsto \varphi_+$$

$$\psi|_A \longleftarrow\!\shortmid\ \psi$$

<u>Prop/exer 2.</u> A = Ban. alg. Then

(1) A_+ is a Ban. alg. w.r.t. $\|a+\lambda 1_+\| = \|a\| + |\lambda|$.

(2) Prop 1 holds for Ban algebras with "Hom" = cont. algebra homom.

<u>Cor.</u> A = Ban. alg, $\chi : A \to \mathbb{C}$ char $\Longrightarrow$ χ is cont, and $\|\chi\| \leq 1$.

<u>Example</u> X = loc. comp. Hausd top. space

X_+ = the 1-point compactification of X

$X_+ = X \sqcup \{\infty\}$.

Topol. on X_+: $\{U \subset X : U \text{ is open}\} \cup \{X_+ \setminus K : K \subset X \text{ comp}\}$

<u>Facts.</u> (1) X_+ is comp. and Hausdorff

(2) Y = comp Hausd top. space; $X = Y \setminus \{y_0\}$, then X is loc. comp, and $\exists$ a homeo $X_+ \xrightarrow{\sim} Y$,

$x \in X \mapsto x \in X$, $\infty \mapsto y_0$.

Exer. (1) $C_0(X) = \{ f|_X : f \in C(X_+),\ f(\infty) = 0 \}$.

(2) $\exists$ a top algebra isomorphism

$$C_0(X)_+ \xrightarrow{\sim} C(X_+), \quad f + \lambda 1_+ \mapsto f + \lambda \quad (f(\infty) = 0).$$

A = algebra, $a \in A$.

Def The nonunital spectrum of a is

$$\sigma_A'(a) = \sigma_{A_+}(a).$$

<u>Observe</u>: $A \subset A_+$ is a 2-sided ideal

$\Rightarrow a \in A$ is not invertible in $A_+ \Longrightarrow 0 \in \sigma'_A(a)$

<u>Exer.</u> (1) A_1, A_2 = unital algebras, $a = (a_1, a_2) \in A_1 \oplus A_2$

$\Longrightarrow \sigma_A(a) = \sigma_{A_1}(a_1) \cup \sigma_{A_2}(a_2)$

(2) A = unital alg $\Longrightarrow \exists$ an algebra isom.

$$A \oplus \mathbb{C} \xrightarrow{\sim} A_+ \ , \ (a, \lambda) \longmapsto a + \lambda(1_+ - 1_A)$$

(3) A = unital alg, $a \in A \Longrightarrow \sigma'_A(a) = \sigma_A(a) \cup \{0\}$.

$A =$ Ban. alg $a \in A$.

<u>Def</u> The <u>spectral radius</u> of a is

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma_A(a)\}.$$

<u>Thm.</u> $r(a) = \lim\limits_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \inf\limits_{n\geq 1} \|a^n\|^{\frac{1}{n}}$.

Max and Γ for nonunital comm. Ban. algebras

A = comm. alg.

Def An ideal $I \subset A$ is <u>modular</u> (<u>regular</u>) if A/I is unital.

$\iff \exists u \in A$ s.t. $\forall a \in A$ $a - au \in I$.

(a <u>modular identity</u> for I).

Observations. (1) $0 \subset A$ is modular $\iff A$ is unital

$\iff$ all ideals of A are modular.

(2) $I \subset J \subset A$ ideals, I is modular $\Rightarrow$ J is modular.

(3) $\chi : A \to \mathbb{C}$ char $\Rightarrow$ $\operatorname{Ker}\chi$ is a mod ideal

(4) Let $A^2 = \operatorname{span}\{ab : a, b \in A\}$. Suppose $A^2 \neq A$.

Then each vec. subspace I s.t. $A^2 \subset I \subsetneq A$ is a non-modular ideal of A.

For ex, $A = t\mathbb{C}[t]$, $I = A^2 = t^2\mathbb{C}[t]$.

Def. The *max spectrum* of A is
$$\operatorname{Max} A = \{\text{max. modular ideals of } A\}.$$

Thm. Each proper modular ideal of A is contained in a max. modular ideal.

Proof: exer.

Exer. Fails for non-modular ideals!

Def The <u>character space</u> of A is
$$\hat{A} = \{\chi : A \to \mathbb{C} : \chi \text{ is char}, \chi \neq 0\}.$$

Exer. The map $\hat{A} \to \operatorname{Max} A$, $\chi \mapsto \operatorname{Ker} \chi$, is injective.

Notation $\hat{A}_+ = \{\text{all characters } A \to \mathbb{C}\} = \hat{A} \cup \{0 : A \to \mathbb{C}\}.$

$$\operatorname{Max}_+ A = \operatorname{Max} A \cup \{A\}.$$

Prop.

$$\begin{tikzcd} \widehat{A_+} \arrow[r, "x \mapsto x|_A"] \arrow[d] \arrow[dr, phantom, "\mathrm{D}"] & \hat{A}_+ \arrow[d] \\ \mathrm{Max}(A_+) \arrow[r, "I \mapsto I \cap A"'] & \mathrm{Max}_+ A \end{tikzcd} \qquad \begin{tikzcd} x \arrow[d, mapsto] \\ \mathrm{Ker}\, x \end{tikzcd}$$

The diag. commutes, and the horiz. arrows are bijections.

Proof: exer.

Hint. $I \subset A$ mod. ideal, $u \in A$ a mod. identity for I. Define $J = I \oplus \mathbb{C}(1_+ - u)$. Then J is an ideal of A_+, and $A_+/J \cong A/I$.

Cor. A = comm Ban alg. Then

(1) All arrows in (D) are bijections

(2) All max. modular ideals of A are closed in A

(3) The map $\hat{A} \to \operatorname{Max} A$, $\chi \mapsto \operatorname{Ker} \chi$, is bijective

Def The <u>Gelfand topology</u> on $\operatorname{Max} A \cong \hat{A}$ and on $\operatorname{Max}_+ A \cong \hat{A}_+$ is the restr. of the weak* top on A^*.

Prop. $\operatorname{Max} A$ and $\operatorname{Max}_+ A$ are Hausdorff, $\operatorname{Max}_+ A$ is compact and $\operatorname{Max}_+ A \underset{\text{homeo}}{\cong} \operatorname{Max}(A_+)$; $\operatorname{Max} A$ is loc. compact, and $\operatorname{Max}_+ A$ is the 1-point compactification of $\operatorname{Max} A$.

$A =$ comm. Ban. alg.

<u>Def</u> The <u>Gelfand transform</u> of $a \in A$ is $\hat{a} : \underset{\hat{A}}{\overset{\text{is}}{\operatorname{Max} A}} \to \mathbb{C}$,

$\hat{a}(\chi) = \chi(a) \quad (\chi \in \hat{A})$.

<u>Prop</u> $\hat{a} \in C_0(\operatorname{Max} A)$.

<u>Proof</u> Extend $\hat{a}$ to $\hat{a} : \operatorname{Max}_+ A \cong \hat{A}_+ \to \mathbb{C}$,

$\hat{a}(\chi) = \chi(a)$.

$\hat{a}$ is cont on $\hat{A}_+$ (see the unital case)

$\hat{a}(0) = 0 \implies \hat{a} \in C_0(\hat{A})$. $\square$

Def The Gelfand transform of A is

$$\Gamma_A : A \longrightarrow C_0(\mathrm{Max}\,A), \quad a \longmapsto \hat{a}.$$

Observe: the diag. commutes

$$\begin{tikzcd} A \arrow[rr, "\Gamma_A"] \arrow[d, phantom, "\cap"] & & C_0(\mathrm{Max}\,A) \arrow[d, phantom, "\cap"] \\ A_+ \arrow[r, "\Gamma_{A_+}"'] & C(\mathrm{Max}(A_+)) \cong & C(\mathrm{Max}_+ A) \end{tikzcd}$$

Thm. (1) Γ_A is an alg hom.

(2) $\|\Gamma_A\| \leq 1$

(3) $\forall a \in A \quad \|\hat{a}\|_\infty = r(a)$.

(4) $\forall a \in A \quad \sigma'_A(a) = \hat{a}(\mathrm{Max}A) \cup \{0\}$.

(5) $\mathrm{Ker}\,\Gamma_A = \bigcap \{\text{max. modular Ideals of } A\}$

$= \{\text{quasinilpotents of } A\}$.

Products and unitizations of C^*-algebras

1. Products.

Observe: (1) $A, B =$ Ban $*$-algebras $\Longrightarrow$ so is $A \times B \cong A \oplus B$:

$$(a,b)^* = (a^*, b^*);$$

$$\|(a,b)\| = \max\{\|a\|, \|b\|\}.$$

(2) $A, B = C^*$-alg $\Longrightarrow$ so is $A \oplus B$

2. Unitizations.

Observe: if A is a Ban. $*$-alg, then so is A_+:

$$(a+\lambda 1_+)^* = a^* + \overline{\lambda} 1_+ \quad (a \in A, \lambda \in \mathbb{C})$$

$$\|a+\lambda 1_+\| = \|a\| + |\lambda|. \qquad (1)$$

Exer. If $A \neq 0$ is a C^*-alg, then norm (1) does not satisfy the C^*-axiom.

Suppose A is a unital C^*-alg.

$A_+ \cong A \oplus \mathbb{C}$ (algebra isom) $\quad (a,\lambda) \in A \oplus \mathbb{C} \mapsto a + \lambda(1_+ - 1_A)$

Hence A_+ becomes a C^*-alg w.r.t.

$$\|a+\lambda(1_+-1_A)\| = \max\{\|a\|, |\lambda|\}.$$

Equivalently

$$\|a+\lambda 1_+\| = \max\{\|a+\lambda 1_A\|, |\lambda|\}.$$

Prop. $A =$ (strictly) nonunital C^*-alg.

$\forall a \in A_+$ let $L_a : A \to A$, $L_a(b) = ab$.

Define $\|a\|_+ = \|L_a\| = \sup\{\|ab\| : \|b\| \leq 1, b \in A\}$.

Then

(1) $\|\cdot\|_+$ is a norm on A_+

(2) $\forall a \in A$ $\|a\|_+ = \|a\|$

(3) $(A_+, \|\cdot\|_+)$ is a C^*-alg

Proof (2) $\forall b \in A \quad \|ab\| \le \|a\|\,\|b\| \implies \|a\|_+ \le \|a\|.$

$\|aa^*\| = \|a\|^2 = \|a\|\,\|a^*\| \implies \|a\|_+ = \|a\|.$

(1) Clearly, $\|\cdot\|_+$ is a seminorm

Suppose $a \in A_+$, $a \ne 0$, $\|a\|_+ = 0$ (that is, $L_a = 0$)

$a = b + \lambda 1_+$. By (2), $\lambda \ne 0$.

$\forall c \in A \quad 0 = ac = bc + \lambda c \implies (-\lambda^{-1} b)c = c,$

that is, $e = -\lambda^{-1} b$ is a left identity in A

$\implies e^*$ is a right id in $A \implies A$ is unital, a contr.

Lemma 1 E = normed sp, $E_0 \subset E$ vec. subspace of codim 1. If E_0 is complete, then so is E.

Lemma 2. A = Ban alg (equipped with an invol) s.t. $\forall a \in A$ $\|a\|^2 \le \|a^* a\| \Longrightarrow A$ is a C^*-alg.

} EXER

Proof of (3). By L1, A_+ is a Ban. alg.

$\forall a \in A_+$ $\forall b \in A$

$$\|ab\|^2 = \|(ab)^* ab\| = \|b^* a^* a b\| \le \|b^*\| \, \|a^* a b\| \le$$

$$\leq \|b^*\| \, \|a^*a\|_+ \|b\| = \|a^*a\|_+ \|b\|^2.$$

$$\Longrightarrow \|a\|_+^2 \leq \|a^*a\|_+ \xrightarrow[L2]{} A_+ \text{ is a } C^*\text{-alg.} \quad \square$$