Spectral radius

A = unital Ban. alg, $a \in A$ $\quad (A \neq 0)$.

Def The spectral radius of a is

$$r_A(a) = \sup\{|\lambda| : \lambda \in \sigma_A(a)\}.$$

Gelfand' thm $\Rightarrow r_A(a) \le \|a\|$.

Example $A = \ell^\infty(X) \Rightarrow r_A(a) = \|a\|$.

The same holds for $C_b(X)$, X = top. space.

Example $A = \mathcal{B}(H)$ H = fin-dim Hilb. space.

$a \in A$ $\quad a = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ w.r.t. an orthonormal basis

$$\Rightarrow r_A(a) = \max_{1 \le i \le n} |\lambda_i| = \|a\|.$$

Example $A = \mathcal{B}(\mathbb{C}^2)$ $\quad a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow r_A(a) = 0$, but $\|a\| > 0$.

Exer. $a \in A$ is nilpotent $\Rightarrow \sigma_A(a) = \{0\}$.

$\Rightarrow r_A(a) = 0$.

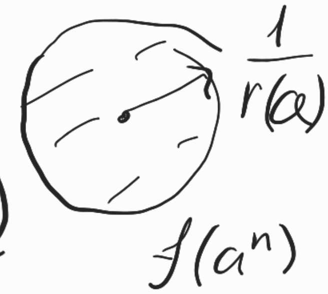
Thm. (Beurling, Gelfand)

A = unital Ban. alg, $a \in A \implies$

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \inf_{n\geq 1} \|a^n\|^{1/n}$$

Idea: $\leq$ triv; $|\lambda^n| \leq \|a^n\|$

$(\geq)$ $f \in A^*$; $\lambda \mapsto f((1-\lambda a)^{-1})$



Cor. $r(a) = 0 \iff \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = 0 \iff$

$\iff \forall\, \varepsilon > 0 \quad \|a^n\| = o(\varepsilon^n) \quad (n \to \infty)$

Def Such elements are called <u>quasinilpotent</u>.

Example/exer.

The Volterra integral operator

$$V_K : L^2[a,b] \to L^2[a,b], \quad (V_K f)(x) = \int_a^x K(x,y) f(y)\, dy$$

is quasinilp. $\forall$ bdd measurable

K on $[a,b] \times [a,b]$.

Cor. 2. A = unital Ban. alg, $B \subset A$ closed subalg, $1_A \in B$. $\implies$

$\implies \forall\, b \in B \quad r_B(b) = r_A(b)$.

The maximal spectrum and the Gelfand transform

A = commutative unital algebra.

Def. An ideal $I \subsetneq A$ is <u>maximal</u> $\iff$
$\nexists$ ideal J s.t. $I \subsetneq J \subsetneq A$.

Exer. I is maximal $\iff A/I$ is a field.

Def. The <u>maximal spectrum</u> of A is
$\mathrm{Max}(A) = \{$maximal ideals of $A\}$.

Exer/example

$$\mathrm{Max}\,\mathbb{C}[t] \overset{1-1}{\rightleftarrows} \mathbb{C}$$

$$p \in \mathbb{C} \longmapsto m_p = \{ f \in \mathbb{C}[t] : f(p) = 0 \}.$$

Prop. Each proper ideal of A is contained in a max. ideal.

Proof. $I \subsetneq A$ ideal

$M = \{ J : J \subsetneq A$ is an ideal, $I \subset J \}$

Claim: $(M, \subset)$ satisfies the conditions of Zorn's lemma.

Indeed: suppose $C \subset M$ is a chain.

Let $K = \bigcup\{J : J \in C\}$

K is an ideal, $I \subset K$.

$\forall J \in C \quad 1 \notin J \Rightarrow 1 \notin K \Rightarrow K \neq A$.

K is an upper bound for $C \Rightarrow$

$\Rightarrow M$ has a max. element. $\square$

<u>Def.</u> The <u>character space</u> of A is

$\hat{A} = \{\chi : A \to \mathbb{C} : \chi \text{ is a character}, \chi \neq 0\}$.

<u>Observe</u>: $\forall \chi \in \hat{A} \quad \operatorname{Ker}\chi \in \operatorname{Max}(A)$

<u>Prop</u> The map $\hat{A} \to \operatorname{Max}(A)$, $\chi \mapsto \operatorname{Ker}\chi$, is injective.

<u>Proof</u> $\chi_1, \chi_2 \in \hat{A}$; $\operatorname{Ker}\chi_1 = \operatorname{Ker}\chi_2 \Rightarrow$

$\Rightarrow \chi_1 = \lambda\chi_2 \quad (\lambda \in \mathbb{C})$; $\quad 1 = \chi_1(1) = \lambda\chi_2(1) = \lambda$

$\Rightarrow \chi_1 = \chi_2$. $\square$

<u>Example/exer.</u>

(1) $A = \mathbb{C}[t] \Rightarrow \hat{A} \to \operatorname{Max}(A)$ is a bijection.

(2) $A \supsetneq \mathbb{C}$ is a field $\Rightarrow \hat{A} = \emptyset$, but $\operatorname{Max}A = \{0\}$.

Lemma. A = comm. unital Ban. alg $\Rightarrow$
$\Rightarrow$ each max. ideal of A is closed in A.

Proof. Let $I \in \mathrm{Max}(A)$. $\Rightarrow \overline{I}$ is an ideal.
Suppose $I \neq \overline{I} \Rightarrow \overline{I} = A \Rightarrow$
$\Rightarrow I \cap A^\times \neq \emptyset$ (because $A^\times$ is open in A)
$\Rightarrow I = A$, a contradiction $\square$

Cor. A comm. unital Ban. alg does not have dense proper ideals.

Thm. A = comm. unital Ban. alg $\Rightarrow$
$\Rightarrow$ the map $\widehat{A} \to \mathrm{Max}(A)$, $\chi \mapsto \operatorname{Ker}\chi$,
is a bijection.

Observation: A = Ban. alg, $I \subset A$ closed 2-sided ideal of $A \Rightarrow A/I$ is a Ban. alg. w.r.t. $\|a+I\| = \inf\{\|a+b\| : b \in I\}$.

Proof of Thm Let $I \in \mathrm{Max}(A) \Rightarrow$
$\Rightarrow A/I$ is a Ban. field $\Rightarrow A/I \cong \mathbb{C}$

$$\begin{tikzcd} A \arrow[r, "\text{quot.}"] \arrow[rr, bend right, "\chi"'] & A/I \arrow[r, "\cong"] & \mathbb{C} \end{tikzcd}$$

$I = \operatorname{Ker}\chi$. $\square$

Cor. A = comm. unital Ban. alg, $a \in A$.

$a \in A^\times \iff \forall \chi \in \hat{A} \quad \chi(a) \neq 0$.

Proof. $(\Rightarrow)$ clear.

$(\Leftarrow)$ Suppose $a \notin A^\times \Rightarrow Aa \subsetneq A \Rightarrow$

$\Rightarrow \exists\ I \in \mathrm{Max}(A)$ s.t. $I \supset Aa$; but $I = \operatorname{Ker}\chi$

$(\chi \in \hat{A}) \Rightarrow \chi(a) = 0$. $\square$

Convention Identify $\hat{A}$ with $\mathrm{Max}(A)$.

Some facts on the weak* topology

E = normed space.

$\forall v \in E$ define a seminorm $\|\cdot\|_v$ on E^*

by $\|f\|_v = |f(v)|$.

Def. The *weak* topology* on E^* is the

loc. convex topology gen. by $\{\|\cdot\|_v : v \in E\}$.

Explicitly: $\forall f \in E^*$ the standard subbase

of nbhds of f (for wk*) is

$\mathfrak{S}_f = \{U_{v,\varepsilon}(f) : v \in E, \varepsilon > 0\}$, where

$$U_{v,\varepsilon}(f) = \{g \in E^* : |g(v) - f(v)| < \varepsilon\}.$$

Facts: (0) wk* is Hausdorff.

(1) $(E^*, wk^*) \subset \mathbb{C}^E$

wk* = the restriction to E^* of the product (Tychonoff) topology on $\mathbb{C}^E$.

(2) $f_n \to f$ w.r.t wk* $\Longleftrightarrow f_n(v) \to f(v) \ \forall v \in E$.

(3) $\forall v \in E$ let $\mathcal{E}_v : E^* \to \mathbb{C}$, $\mathcal{E}_v(f) = f(v)$.

wk* = the weakest topology on E^* that makes all $\mathcal{E}_v$ continuous.

(4) X = top. space

A map $\varphi : X \to (E^*, wk^*)$ is cont $\Longleftrightarrow$

$\Longleftrightarrow \forall v \in E \ \ \mathcal{E}_v \circ \varphi : X \to \mathbb{C}$ is cont.

(5) $i_E : E \to E^{**}$ can. emb. $(v \mapsto \mathcal{E}_v)$.

$\operatorname{Im} i_E = \{\alpha \in E^{**} : \alpha \text{ is } wk^*\text{-cont}\}$.

(6) E, F normed

A lin. oper. $T : (F^*, wk^*) \to (E^*, wk^*)$ is cont $\Longleftrightarrow \exists$ a bdd lin. op. $S : E \to F$ st $S^* = T$.

(7) (Banach-Alaoglu Thm)

$\mathbb{B}_{E^*} = \{f \in E^* : \|f\| \le 1\}$ is wk*-compact.

A = comm. unital Banach alg.

<u>Def</u> The <u>Gelfand topology</u> on $\mathrm{Max}(A) \cong \hat{A}$ is the restriction to $\hat{A}$ of the weak* top. on A^*.

$$\mathrm{Max}(A) \cong \hat{A} \subset A^*$$

<u>Thm.</u> $\mathrm{Max}(A)$ is compact and Hausdorff.

<u>Proof</u> (A^*, wk^*) is Hausd $\Rightarrow$ so is $\hat{A}$.

$\hat{A} \subset \mathbb{B}_{A^*}$, $(\mathbb{B}_{A^*}, wk^*)$ is compact.

We have to show that $\hat{A} \subset \mathbb{B}_{A^*}$ is closed.

Let $a, b \in A$. Observe: the maps $A^* \to \mathbb{C}$

$$\left.\begin{aligned} f \in A^* &\mapsto f(ab) - f(a)f(b) \\ f \in A^* &\mapsto f(1) \end{aligned}\right\} \text{ are cont wrt } wk^*$$

$$\hat{A} = \left\{ f \in A^* : \begin{aligned} &f(ab) - f(a)f(b) = 0 \ \forall a, b \in A; \\ &f(1) = 1 \end{aligned} \right\}.$$

$\Rightarrow \hat{A}$ is closed in $\mathbb{B}_{A^*}$. $\square$

<u>Def</u> The <u>Gelfand transform</u> of $a \in A$ is

$$\hat{a} : \mathrm{Max}(A) \to \mathbb{C}, \quad \hat{a}(x) = x(a).$$

<u>Prop</u> $\hat{a}$ is continuous.

<u>Proof</u> $\hat{a} = i_A(a)|_{\hat{A}}$; $i_A(a)$ is wk^*-cont on A^*. $\square$

Def. The <u>Gelfand transform</u> of A is
$\Gamma_A : A \to C(\mathrm{Max}\,A)$, $a \in A \mapsto \hat{a}$.

Thm. (properties of Γ_A)
A = comm. unital Ban. algebra.
(1) Γ_A is a unital algebra homom.
(2) $\|\Gamma_A\| = 1$ (if $A \neq 0$)
(3) $\forall a \in A$ $\|\hat{a}\|_\infty = r_A(a)$
(4) $\forall a \in A$ $\sigma_A(a) = \hat{a}(\mathrm{Max}\,A)$.
(5) $\mathrm{Ker}\,\Gamma_A = \bigcap \{I : I \in \mathrm{Max}\,A\} =$
$= \{a \in A : a \text{ is quasinilp.}\}$

Proof (1) exer.
(4) We know: $\hat{a}(\mathrm{Max}A) = \sigma_{C(\mathrm{Max}\,A)}(\hat{a})$
$\Longrightarrow$ it suff. to show that
$\Gamma(\text{noninv}) \subset \text{noninv}$.
Suppose $a \notin A^\times \Rightarrow \exists\, \chi \in \hat{A}$ s.t. $\chi(a) = 0$,
that is, $\hat{a}(\chi) = 0$ $\Longrightarrow$ $\hat{a}$ is noninv in $C(\mathrm{Max}A)$
(3) follows from (4)
(2) $\forall a \in A$ $\|\hat{a}\|_\infty = r(a) \le \|a\|$ $\Longrightarrow$ $\|\Gamma_A\| \le 1$;

$\Gamma_A(1) = 1 \implies \|\Gamma_A\| = 1.$

(5) $\operatorname{Ker}\Gamma_A = \bigcap\{\operatorname{Ker}\chi : \chi \in \hat{A}\} =$

$= \bigcap\{I : I \in \operatorname{Max}(A)\} \underset{(3)}{=} \{\text{quasinilpotents}\}.$

$\square$

<u>Def</u> A = unital comm. alg.

The <u>Jacobson radical</u> of A is

$J(A) = \bigcap\{I : I \in \operatorname{Max}(A)\}.$

A is <u>Jacobson semisimple</u> $\iff J(A) = 0.$

<u>Cor.</u> $\operatorname{Im}\Gamma_A$ is spec. invariant in $C(\operatorname{Max}A).$

<u>Proof.</u> $\Gamma(a) \in C(\operatorname{Max}A)^\times \implies a \in A^\times \implies$

$\implies \Gamma(a) \in (\operatorname{Im}\Gamma_A)^\times.$ $\square$

Examples: subalgebras of $C(X)$

X = compact Hausd. top. space

$\forall x \in X \quad \varepsilon_x : C(X) \to \mathbb{C}, \quad \varepsilon_x(f) = f(x)$;

$m_x = \operatorname{Ker} \varepsilon_x$.

<u>Lemma</u> $\forall$ ideal $I \subsetneq C(X)$ $\exists x \in X$ s.t. $I \subset m_x$

<u>Proof</u>. Suppose $\forall x \in X$ $\exists f_x \in I$ s.t. $f_x(x) \neq 0$.

$\exists$ a nbhd $U_x \ni x$ s.t. $\forall y \in U_x \quad f_x(y) \neq 0$.

$X = U_{x_1} \cup \ldots \cup U_{x_n}$ (by compactness)

Let $f = \sum_{i=1}^{n} |f_{x_i}|^2 = \sum_{i=1}^{n} \bar{f}_{x_i} f_{x_i} \in I$;

$f(y) > 0 \ \forall y \in X \Rightarrow f$ is invertible in $C(X)$

$\Rightarrow I = C(X)$, a contr. $\square$.

Cor The map $\varepsilon : X \to \operatorname{Max} C(X)$, $x \mapsto m_x$

is a bijection. $\qquad \searrow \widehat{C(X)}$, $x \mapsto \varepsilon_x$.

Notation. X, Y compact, Hausd.

$f : X \to Y$ cont.

$f^\bullet : C(Y) \to C(X)$, $\quad f^\bullet(\varphi) = \varphi \circ f$.

Properties of $f^\bullet$:

(1) $f^\bullet$ is a unital alg. hom, and $\|f^\bullet\| = 1$.

(2) $(1_X)^\bullet = 1_{C(X)}$

(3) $X \xrightarrow{f} Y \xrightarrow{g} Z \implies (g \circ f)^\bullet = f^\bullet \circ g^\bullet$.

Observe: (1)-(3) $\implies$ if f is a homeo, then $f^\bullet$ is an isometric isomorphism.

Thm. X = compact Hausd. top. space

$A \subset C(X)$ subalg, $1_{C(X)} \in A$. Suppose

(1) A is a Ban. alg. w.r.t a norm that dominates the sup norm.

(2) A separates the points of X

(3) $\forall \chi \in \hat{A}$ $\exists x \in X$ s.t. $\chi = \varepsilon_x$.

Then the map $\varepsilon : X \to \underline{\hat{A}}$, $x \mapsto \varepsilon_x$, is a homeomorphism. Moreover, the foll. diag. commutes:

$$\begin{tikzcd} & C(X) \\ \underline{A} \arrow[ur, hook] \arrow[dr, "\Gamma_A"'] & \\ & C(\operatorname{Max} A) \arrow[u, "\varepsilon^\circ", "\wr"'] \end{tikzcd}$$

Proof. (2) & (3) $\Rightarrow$ ε is a bijection

ε is cont $\iff$ $\forall a \in A$ the map $x \mapsto \underset{a(x)}{\underset{\parallel}{\varepsilon(x)(a)}}$ is cont.

$a \in C(X) \Rightarrow \varepsilon$ is cont $\Rightarrow$ ε is a homeo.

$$(\varepsilon^\circ \Gamma)(a)(x) = \Gamma(a)(\varepsilon_x) = \varepsilon_x(a) = a(x)$$

$\Rightarrow$ the diag. commutes. $\square$

Cor. If $A = C(X)$ (X compact, Hausd)

$\Rightarrow \Gamma_A$ is an isometric isomorphism, and $\Gamma_A^{-1} = \varepsilon^\bullet$.

Functorial properties of Γ

Comp — objects: comp. Hausd. top spaces; morphisms: cont. maps.

CUBA — objects: comm. unital Ban. alg.; morphisms: cont. unital homs.

2 contravar. functors:

$C : \text{Comp} \to \text{CUBA}$, $X \mapsto C(X)$;

$(f : X \to Y) \mapsto \left(f^\bullet : C(Y) \to C(X),\ f^\bullet(\varphi) = \varphi \circ f \right)$

$\text{Max} : \text{CUBA} \to \text{Comp}$, $A \mapsto \text{Max}(A)$

$(\varphi : A \to B) \mapsto \left(\varphi^* : \text{Max}(B) \to \text{Max}(A),\ \varphi^*(\chi) = \chi \circ \varphi \right)$.

φ^* is the restr. of $\varphi^* : B^* \to A^*$ (dual of φ), which is wk*-cont $\Rightarrow \varphi^* : \text{Max} B \to \text{Max} A$ is cont.

Exer.

(1) $\{\varepsilon_X : X \to Max(C(X)) : X \in Comp\}$

is a natural isom. btw 1_{Comp} and $Max \circ C$.

(2) $\{\Gamma_A : A \to C(Max(A)) : A \in CUBA\}$

is a natural transformation from 1_{CUBA} to $C \circ Max$.

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\Gamma_A"'] & B \arrow[d, "\Gamma_B"] \\ C(Max A) \arrow[r] & C(Max B) \end{tikzcd}$$

(3) $\exists$ 1-1 correspondence

$$Hom_{CUBA}(A, C(X)) \cong Hom_{Comp}(X, Max A) \cong Hom_{Comp^{op}}(Max A, X)$$

$$\varphi \longmapsto \varphi^* \circ \varepsilon_X$$

$$f^{\circ} \circ \Gamma_A \longleftarrow\!\shortmid\; f$$

Hence (Max, C) is an adjont pair of functors.