$$L^1(G) \hookrightarrow M(G)$$

$$f \longmapsto f \cdot (\text{Haar})$$

δ_e

Approximate identities

$(\Lambda, \leq)$ poset

Def $(\Lambda, \leq)$ is <u>directed</u> if $\forall \lambda, \mu \in \Lambda \; \exists \nu \in \Lambda$ s.t. $\lambda \leq \nu$, $\mu \leq \nu$.

Examples (1) $(\mathbb{N}, \leq)$

(2) $X =$ top space, $x \in X$.

$\Lambda = \{\text{neighborhoods of } x\}$. $(\Lambda, \supset)$ is a dir. poset.

X = top space.

<u>Def</u> A <u>net</u> in X is a map $x : \Lambda \to X$, where Λ is a dir. poset.
(= направленность)

<u>Notation</u> $x_\lambda = x(\lambda)$ $\qquad x = (x_\lambda)_{\lambda \in \Lambda}$.

<u>Def</u> (x_λ) <u>converges</u> to $x \in X$ $\left(x_\lambda \xrightarrow[\Lambda]{} x;\ \lim\limits_{\Lambda} x_\lambda = x \right)$ if
$\forall$ nbhd $U \ni x$ $\exists \lambda_0 \in \Lambda$ s.t. $\forall \lambda \geq \lambda_0$ $x_\lambda \in U$.

<u>Example</u> Λ = the poset from Ex(2)
$\forall U \in \Lambda$ choose $x_U \in U$. $\qquad x_U \to x$.

A = normed alg.

Def An <u>approximate identity</u> (a.i.) in A is a net (e_λ) in A s.t. $\forall a \in A$ $\quad a e_\lambda \to a, \quad e_\lambda a \to a$.

Def (1) An a.i. $(e_\lambda)_{\lambda \in \Lambda}$ is <u>sequential</u> if $\Lambda = \mathbb{N}$ with the standard order.

(2) $(e_\lambda)_{\lambda \in \Lambda}$ is a <u>bounded</u> appr. id. if $\exists C > 0$ s.t.

$$\|e_\lambda\| \leq C \ \forall \lambda.$$

(b.a.i. = bounded a.i.)

Example 1 $A = c_0 = C_0(\mathbb{N}) = \{x = (x_n) \in \mathbb{C}^{\mathbb{N}} ; \lim_{n\to\infty} x_n = 0\}$.

$\forall n \in \mathbb{N}$ $e_n = (\underbrace{1, \dots, 1}_{n}, 0, 0 \dots) \in A$ $\|e_n\| = 1$.

$\forall a \in A$

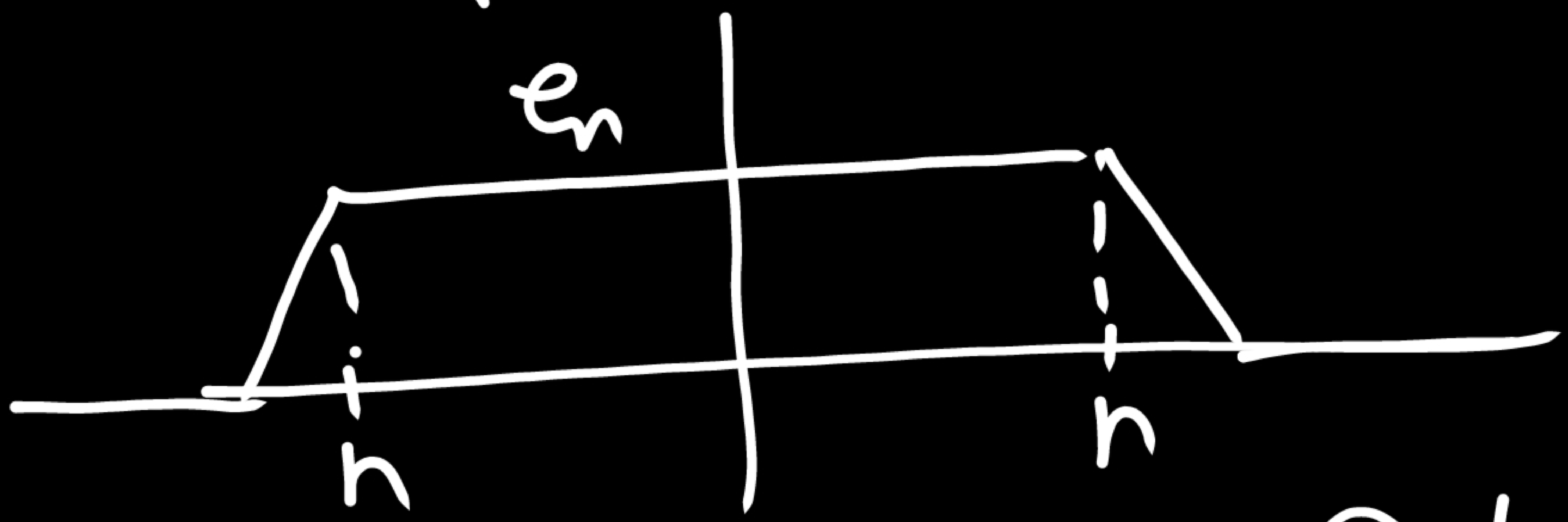
$$\|a - ae_n\| = \sup_{k>n} |a_k| \to 0 \implies (e_n) \text{ is a b.a.i in } A.$$

Example 2 $A = \ell^1$ with pointwise mult.

$\Rightarrow (e_n)_{n\in\mathbb{N}}$ is an unbounded a.i.

Exer ℓ^1 does not have a b.a.i.

Example 3. $A = C_0(\mathbb{R})$



(e_n) is a b.a.i. in $C_0(\mathbb{R})$.

Example 4. $A = C_0(X)$ (X = loc comp. Hausd. space).

$\Lambda = \{K \subset X : K \text{ is comp}\}$. $(\Lambda, \subset)$ is a dir. poset.

$\forall K \in \Lambda$ choose $e_K \in C_0(X)$ st. $e_K|_K = 1$, $\|e_K\| \leq 1$

Exer: $(e_K)_{K \in \Lambda}$ is a b.a.i in $C_0(X)$.

Exer. $C_0(X)$ has a sequential b.a.i. $\Longleftrightarrow$ X is σ-comp.

Example 5. $A = \mathcal{K}(H)$ (H = Hilb. space).

$\Lambda = \{ L \subset H : L \text{ is a fin-dim vec subspace} \}$.

$(\Lambda, \subset)$ is a dir. poset

$\forall L \in \Lambda$ let P_L = the orth proj. onto L.

Exer. $(P_L)_{L \in \Lambda}$ is a b.a.i. in $\mathcal{K}(H)$.

Exer. $\mathcal{K}(H)$ has a sequential b.a.i. $\Longleftrightarrow$ H is seperable

Example 6.

(1) $(A, \text{zero mult})$ does not have an a.i.

(2) $A = \{f \in C^1[0,1] : f(0) = 0\}$ does not have an a.i.

Prop / exer. A = normed alg, (e_λ) is a bdd net in A. Suppose $S \subset A$ generates a dense subalg of A and $e_\lambda a \to a$, $a e_\lambda \to a$ $\forall a \in S$. Then (e_λ) is a bai in A

G = loc. comp group (2nd countable) $\qquad \mu$ = Haar meas

β = a base of rel. compact symm. nbhds of $e \in G$.

$\forall V \in \beta$ choose $u_V \in L^1(G)$ s.t.

(1) $u_V \geq 0$;

(2) $u_V|_{G \setminus V} = 0$;

(3) $\int_G u_V \, d\mu = 1$

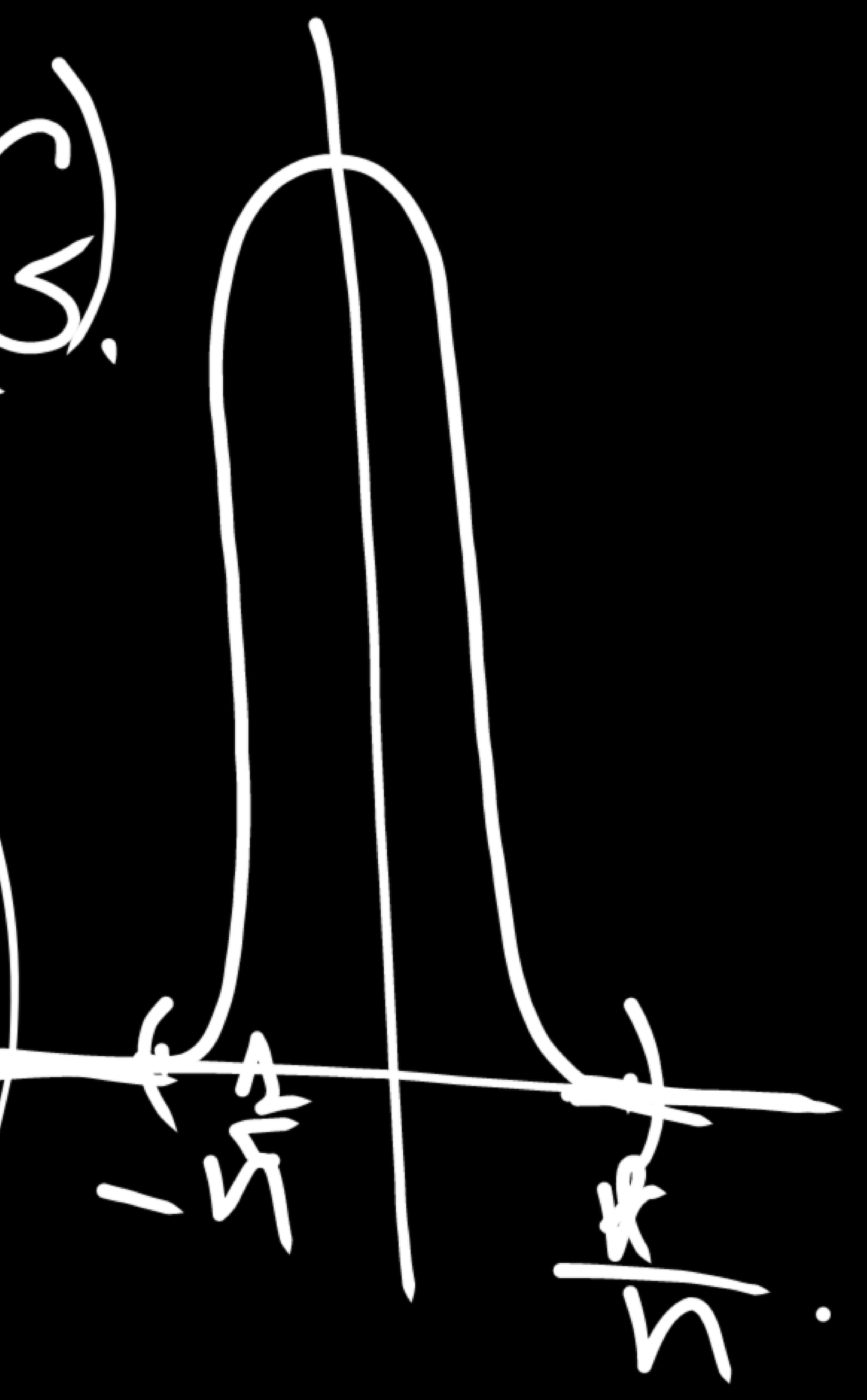
<u>Def</u> A net $(u_V)_{V \in \beta}$ satisfying (1)-(3) is a <u>Dirac net</u> in $L^1(G)$. (<u>δ-образная</u> направленность)

Example $u_V = \dfrac{\chi_V}{\|\chi_V\|_1}$.

Rem. $\exists$ a Dirac net in $C_c(G)$. (Urysohn's lemma).

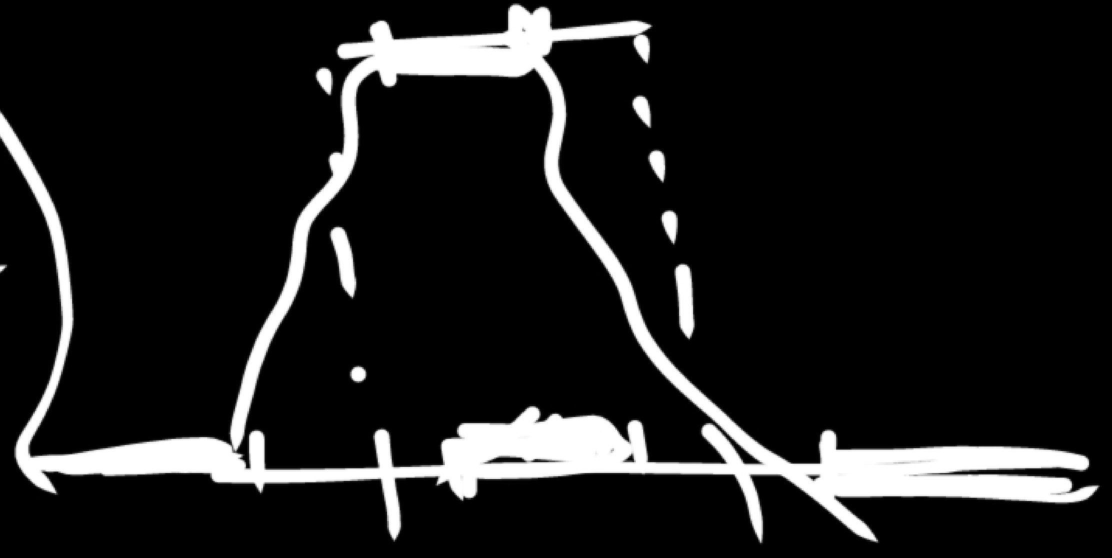
Prop Any Dirac net in $L^1(G)$ is a b.a.i. for $L^1(G)$.

Proof. $C_c(G)$ is dense in $L^1(G)$ (Urysohn's lemma)



Hence it suff to show that

$u_V * f \to f$ and $f * u_V \to f$ $\forall f \in C_c(G)$



We may assume that $\exists V_0 \in \beta$ s.t. $V \subset V_0 \ \forall V \in \beta$.

$$(u_V * f - f)(x) = \int_V u_V(y)\big(f(y^{-1}x) - f(x)\big)\,d\mu(y).$$

$$\|u_V * f - f\|_1 = \int_G \left| \int_V u_V(y)\big(f(y^{-1}x) - f(x)\big)\,d\mu(y) \right| d\mu(x) \leq$$

$$\leq \int_V \int_G |u_V(y)|\,|f(y^{-1}x) - f(x)|\,d\mu(x)\,d\mu(y) =$$

$$= \int_V u_V(y)\,\|L_y f - f\|_1\,d\mu(y) \leq \sup_{y \in V} \|L_y f - f\|_1. \quad (*)$$

Exer. $\exists C>0$ s.t. $\forall y \in V_0$ $\|L_y f - f\|_1 \leq C \|L_y f - f\|_\infty$.

Hence $(*) \leq C \sup_{y \in W} \|L_y f - f\|_\infty \to 0$ by the uniform continuity of f.

- $f * u_V \to f$: exer.

$\square$

Spectral theory in Ban. algebras

(a survey)

A = unital algebra

$A^\times = \{a \in A : a \text{ is invertible}\}$, (mult. group of A)

Def. The *spectrum* of $a \in A$ is

$$\sigma_A(a) = \sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \notin A^\times\}.$$

Example 1. $A = \mathbb{C}$ $\quad \sigma_{\mathbb{C}}(\lambda) = \{\lambda\}$.

Example 2. $A = \mathrm{End}_{\mathbb{C}}(E)$ $\quad \dim E < \infty$.

$\forall T \in A$ $\quad \sigma_A(T) = \{\text{eigenvalues of } T\}$.

Example 3. $A = \mathbb{C}^X$ $\quad$ (X = a set)

$\sigma_A(f) = f(X)$. The same is true for $A = C(X)$ (X = top. space)

...

Example 4. $A = \ell^\infty(X)$ (X = a set)

$$\sigma_A(f) = \overline{f(X)}.$$

The same is true, for example, for $A = C_b(X)$ (X = a top. space).

Example 5. $A = \mathbb{C}G$ (G = a fin. abelian group)

$$\sigma_A(f) = \hat{f}(\hat{G}).$$

Prop. $\varphi : A \to B$ unital alg hom. Then

(1) $\varphi(A^\times) \subset B^\times$.

(2) $\sigma_B(\varphi(a)) \subset \sigma_A(a) \quad \forall a \in A$

(3) $\forall a \in A \quad \sigma_B(\varphi(a)) = \sigma_A(a) \iff \varphi(A \setminus A^\times) \subset B \setminus B^\times$.

Cor. A = unital alg, $B \subset A$ subalg, $1_A \in B$.

Then $\forall b \in B \quad \sigma_A(b) \subset \sigma_B(b)$.

Def B is <u>spectrally invariant</u> in A if $\forall b \in B$
$$\sigma_B(b) = \sigma_A(b) \iff B \setminus B^\times \subset A \setminus A^\times$$
$$\iff B \cap A^\times = B^\times.$$

<u>Examples</u>

(1) $C(X) \subset \mathbb{C}^X$ is spec. inv. (X = a top space)

(2) $\ell^\infty(X) \subset \mathbb{C}^X$ is not spec inv (X = an inf. set)

(3) $\mathcal{B}(E) \subset \mathrm{End}_{\mathbb{C}}(E)$ is spec. inv. (E = a Ban. space)

Prop (polynomial spectral mapping thm)

A = unital alg, $a \in A$, $f \in \mathbb{C}[t]$. Then

$$\boxed{\sigma_A(f(a)) = f(\sigma_A(a))}$$

unless $\sigma_A(a) = \emptyset$ and $f \in \mathbb{C}1$.

Prop. If $a \in A^\times$, then $\sigma(a^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(a)\}$.

<u>Thm</u> A = unital Ban. alg. Then

(1) $A^\times$ is open in A. Moreover:

$\forall a \in A$ s.t. $\|a\| < 1$ $\quad 1 - a \in A^\times$, and

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n$$

$a \in A^\times$

(2) The map $A^\times \to A^\times$, $a \mapsto a^{-1}$, is continuous.

Def. A = an alg.
A **character** of A is an alg hom. $\chi: A \to \mathbb{C}$.

Observe: if A is unital and $\chi \neq 0$, then $\chi(1) = 1$.

Cor. A = unital Ban. alg, $\chi: A \to \mathbb{C}$ a character
$\Rightarrow \chi$ is cont, and $\|\chi\| \leq 1$

Proof If χ is unbdd or $\|\chi\| > 1$, then $\exists\, a \in A$
s.t. $|\chi(a)| > \|a\|$ $\Rightarrow \exists\, b \in A$ st. $\|b\| < 1$, $\chi(b) = 1$
$\left(b = a/\chi(a)\right)$

$\Rightarrow 1-b \in A^\times \Rightarrow \chi(1-b) \neq 0$

$\qquad\qquad\qquad\qquad\quad \overset{\|}{1-\chi(b)} = 0$, a contr. $\square$.

<u>Thm</u> (Gelfand)

A = unital Ban. alg, $a \in A$. Then

(1) $\forall \lambda \in \sigma_A(a) \quad |\lambda| \leq \|a\|$

(2) $\sigma_A(a)$ is compact

(3) $\sigma_A(a) \neq \emptyset$ (if $A \neq 0$)

$f \in A^*$

$\lambda \longmapsto f((a-\lambda 1)^{-1})$

$\mathbb{C} \to \mathbb{C}$

Thm (Gelfand-Mazur thm)

A = a Ban. division alg (that is, $A \neq 0$ and all $a \in A \setminus \{0\}$ are invertible)

Then $A \cong \mathbb{C}$.

Proof $\forall a \in A \ \exists \lambda \in \mathbb{C}$ s.t. $a - \lambda 1 = 0$, that is, $a = \lambda 1$

$\Rightarrow A = \mathbb{C}1 \cong \mathbb{C}$ $\square$