Some operations on measures

$X =$ loc. comp Hausd top space

$C^+(X) = \{f \in C(X) : f \geq 0\}$.

1. Multiplication by a function

Notation. (1) $\forall$ lin. $I : C_c(X) \to \mathbb{C}$, $\forall f \in C(X)$

define $f \cdot I : C_c(X) \to \mathbb{C}$ by $(f \cdot I)(g) = I(fg)$

Observe: if $I, f \geq 0 \implies f \cdot I \geq 0$

(2) $\forall$ Rad. meas μ on X, $\forall f \in C^+(X)$ define a Rad. meas $f\cdot\mu$ on X by $I_{f\cdot\mu} = f\cdot I_\mu$.

<u>Exer</u> (1) If X is σ-compact, then

$$(f\cdot\mu)(B) = \int_B f\,d\mu \quad \forall \text{ Borel } B \subset X.$$

(2) The same is true if $\int_X f\,d\mu < \infty$.

<u>Exer.</u> G = loc. comp group, $f \in C^+(G)$, μ = a Rad meas on $G \Rightarrow L_x(f\cdot\mu) = L_x f \cdot L_x\mu$, $R_x(f\cdot\mu) = R_x f\cdot R_x\mu$ $(x \in G)$

2. Reflection

G = loc. comp group

Notation. (1) $I : C_c(G) \to \mathbb{C}$ linear.

Define $S(I) : C_c(G) \to \mathbb{C}$ by $S(I) = I \circ S$.

(where $(Sf)(x) = f(x^{-1})$ $\forall x \in G$).

(2) $\forall$ Rad. meas μ on G define a Rad meas $S\mu$ on G

by $I_{S\mu} = S(I_\mu)$.

Exer. $(S\mu)(B) = \mu(B^{-1})$ $\forall$ Borel $B \subset G$.

Exer. $S(f \cdot \mu) = Sf \cdot S\mu$ $\forall f \in C^+(G)$

The modular character (modular function)

G = loc. comp. group, μ = a (left) Haar meas on G.

Observe: $\forall x \in G$ $\quad R_x \mu$ is a Haar measure.

Indeed: $L_y(R_x\mu) = R_x L_y \mu = R_x \mu$.

Hence $\exists\ \Delta(x) > 0$ s.t. $\boxed{R_x \mu = \Delta(x)\mu.}$ $(*)$

Def The function $\Delta : G \to \mathbb{R}_{>0}$ given by $(*)$ is called the modular character of G

Prop. 1 $R_x I_\mu = \Delta(x) I_\mu \;\; \forall x \in G.$

That is, $\int_G f(yx)\,d\mu(y) = \Delta(x^{-1}) \int_G f\,d\mu.$

(Recall: $R_x I_\mu = I_\mu \circ R_{x^{-1}}$ by def.)

Proof $R_x I_\mu = I_{R_x \mu} = I_{\Delta(x)\mu} = \Delta(x) I_\mu.$ $\square$

Prop. 2. $\Delta : G \to \mathbb{R}_{>0}$ is a continuous homom.

Proof $\Delta(xy)\mu = R_{xy}\mu = R_x R_y \mu = \Delta(y) R_x \mu =$

$= \Delta(x)\Delta(y)\mu. \implies \Delta$ is a homom;

Choose $f \in C_c(G)$ s.t. $I_\mu f = 1$

$\Longrightarrow \Delta(x) = R_x I_\mu f = I_\mu(R_{x^{-1}} f)$ is continuous (see previous lec). $\square$

Recall: μ = Haar meas $\Longrightarrow S\mu$ is a right Haar meas.

Prop 3. $\boxed{S\mu = \Delta^{-1} \cdot \mu.}$ That is, $\forall f \in C_c(G)$

$$\int_G f(x^{-1})\, d\mu(x) = \int_G \Delta(x)^{-1} f(x)\, d\mu(x)$$

Proof Let $\nu = \Delta^{-1}\cdot\mu$. **Claim**: ν is right invar.

Observe: $\forall$ homom $\varphi : G \to \mathbb{C}^\times$

$$R_x\varphi = \varphi(x)\varphi \;,\quad S\varphi = \varphi^{-1}.$$

$$R_x\nu = R_x(\Delta^{-1}\cdot\mu) = R_x(\Delta^{-1})\cdot R_x\mu = \cancel{\Delta(x)^{-1}}\,\Delta^{-1}\cdot\cancel{\Delta(x)}\mu = \nu.$$

$\Rightarrow \nu$ is right inv $\Rightarrow \exists\, c>0$ s.t. $S\mu = c\cdot\Delta^{-1}\mu \quad (1)$

We want: $c=1$.

$(1) \Rightarrow c\cdot\mu = \Delta\cdot S\mu.$

$(1) \Rightarrow \mu = c\,S(\Delta^{-1}\cdot\mu) = c\cdot S(\Delta^{-1})\cdot S\mu = c\cdot\Delta\cdot S\mu = c^2\mu$

$\Rightarrow c=1$ $\square$

Def. G is <u>unimodular</u> if $\Delta \equiv 1$

$\iff$ a left Haar meas is right inv

$\iff$ a right Haar meas is left inv.

Example 1. Abelian $\Rightarrow$ unimodular.

Example 2 Compact $\Rightarrow$ unimodular.

Indeed: $\Delta(G)$ is a comp subgroup of $\mathbb{R}_{>0}$

$\Longrightarrow \Delta(G) = \{1\}$.

Example/exer 3

$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}$ is not unimodular.

Warning

$\tilde{\Delta}(x) R_x \mu = \mu$

$\tilde{\Delta} = 1/\Delta$.

Exer. G = Lie group ; $\mathfrak{g} = T_e G$.

$\forall x \in G \quad i_x : G \to G \quad i_x(y) = x y x^{-1}$.

$Ad_x = (d i_x)(e) : \mathfrak{g} \to \mathfrak{g}$.

$Ad : G \to GL(\mathfrak{g})$ is a group hom (the adjoint repres. of G)

Prove: $\Delta(x) = |\det Ad_{x^{-1}}|$.

Banach algebras

Def A normed algebra is an alg A equipped with a norm such that $\|ab\| \leq \|a\| \|b\|$ $\forall a,b \in A$.
($\|\cdot\|$ is submultiplicative)
If A is unital, then we require that $\|1_A\| = 1$.

Exer. $A \times A \to A$, $(a,b) \mapsto ab$, is cont.

Def Banach algebra = complete normed alg.

Example 0 $\quad 0, \mathbb{C}$

Example 1. $X =$ a set

$\ell^\infty(X)$ is a Ban alg under pointwise mult.

Example 2 $\quad X =$ top space

$C_b(X) = C(X) \cap \ell^\infty(X)$ is a closed subalg in $\ell^\infty(X)$

$\Longrightarrow$ a Ban. alg.

Def A cont. func $f : X \to \mathbb{C}$ vanishes at ∞ if $\forall \varepsilon > 0$ $\exists$ a comp set $K \subset X$ st. $|f(x)| < \varepsilon$ $\forall x \in X \setminus K$.

Example 3. $C_0(X) = \{ f \in C(X) : f \text{ vanishes at } \infty \}$
is a closed Ideal in $C_b(X)$ $\Longrightarrow$ a Ban. alg
If X is compact, then $C_0(X) = C_b(X) = C(X)$

Example 4 (X, μ) = meas. space
$L^\infty(X, \mu)$ is a Ban. alg under pointwise mult. (exer)

Example 5. $C^n[a,b]$ is Ban. alg w.r.t.
/exer

$$\|f\|_{C^n} = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_\infty}{k!}$$

(equiv. to $\|f\| = \max\{ \|f^{(k)}\| : 0 \le k \le n \}$)

Example 6. $K \subset \mathbb{C}$ comp set

$\mathcal{A}(K) = \{ f \in C(K) : f$ is holom on $\operatorname{Int} K \}$

is a closed subalgebra $\Rightarrow$ a Ban. alg.

$\overline{\mathbb{D}} = \{ z \in \mathbb{C} : |z| \leq 1 \}$.

$\mathcal{A}(\overline{\mathbb{D}})$ is the <u>disc algebra</u>.

Example 7. E = Ban space.

$\mathcal{B}(E) = \{$ bdd linear opers $E \to E \}$ is a Ban alg.

Example 8

$\mathcal{K}(E) = \{T \in \mathcal{B}(E) : T \text{ is compact}\}$ is a closed 2-sided ideal of $\mathcal{B}(E) \Longrightarrow$ a Ban. alg

Def A = an algebra. An involution on A is a map $A \to A$, $a \in A \mapsto a^* \in A$, such that

(1) $a^{**} = a \quad (a \in A)$

(2) $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^* \quad (a, b \in A, \lambda, \mu \in \mathbb{C})$

(3) $(ab)^* = b^* a^*$.

$(A, *)$ is a <u>$*$-algebra</u>.

<u>Def</u> A <u>Banach $*$-alg</u> is a Ban alg A equipped with an invol. such that $\|a^*\| = \|a\|$ $\forall a \in A$.

<u>Def</u> A Ban $*$-alg A is a <u>C^*-algebra</u> if
$\|a^* a\| = \|a\|^2$ $(a \in A)$ (C^*-axiom)

<u>Def.</u> $A, B = *$-alg.
An alg hom. $\varphi: A \to B$ is a <u>$*$-homom</u> if $\varphi(a^*) = \varphi(a)^*$.
$(a \in A)$

Def $A = *$-alg

$S \subset A$ is a *$*$-subset* if $\forall a \in S$ we have $a^* \in S$

(that is, $S^* = S$)

Example. 0, $\mathbb{C}$ are C^*-alg; $\lambda^* = \overline{\lambda}$ $(\lambda \in \mathbb{C})$.

Example. / exer $\underbrace{\ell^\infty(X), C_b(X), C_0(X), L^\infty(X,\mu),}_{C^*\text{-algebras}}$ $C^n[a,b]$

are Ban. $*$-alg w.r.t. $f^*(x) = \overline{f(x)}$.

Exer. $C^n[a,b]$ is not a C^*-alg if $n \geq 1$.

Example/exer. $\mathcal{A}(\overline{\mathbb{D}})$ is a Ban. $*$-alg w.r.t. $f^*(z) = \overline{f(\bar{z})}$ but is not a C^*-alg.

Example H = Hilb space.

$\mathcal{B}(H)$ is a C^*-alg; $\langle T^* x | y \rangle = \langle x | Ty \rangle$.

$\mathcal{K}(H)$ is a closed $*$-ideal in $\mathcal{B}(H)$

$\Rightarrow \mathcal{K}(H)$ is a C^*-alg.

The algebra $L^1(G)$

Prop. 1. X, Y = 2nd countable Hausd loc. comp spaces. Then:

(1) $\mathcal{B}or(X \times Y)$ is gener. by $\{B_1 \times B_2 : B_1 \in Bor(X), B_2 \in Bor(Y)\}$.

(2) μ = a Rad meas on X, ν = a Rad. meas on Y

$\Longrightarrow \mu \otimes \nu$ is a Rad. meas on $X \times Y$.

Proof: exer.

Prop 2. G_1, G_2 = loc. comp groups, 2nd countable.
μ_1, μ_2 = Haar meas on G_1, G_2, resp
Then $\mu_1 \otimes \mu_2$ is a Haar meas on $G_1 \times G_2$.

Proof: exer.

G = loc comp group, μ = Haar meas on G.
$\mathcal{M}_\mu = \{A \subset G : A \text{ is } \mu\text{-measurable}\}$.

Recall: $\mathcal{M}_\mu$ is a σ-alg; $\mathcal{M}_\mu = \left\{ B \cup N : \begin{array}{l} B \subset G \text{ is Borel} \\ N \subset G \text{ is a} \\ \mu\text{-null set} \end{array} \right\}$
(the completion of $\mathrm{Bor}(X)$).

Lemma $f, g : G \to \mathbb{C}$ $\mathcal{M}_\mu$-measurable. Let

$F : G \times G \to \mathbb{C}$, $\quad F(y,x) = f(y)\, g(y^{-1}x)$

Then F is $\mathcal{M}_{\mu\otimes\mu}$-measurable

$L^1(G)$

$$\boxed{(f * g)(x) = \int_G f(y)\, g(y^{-1}x)\, d\mu(y).}$$