Thm 1 G = loc. comp group. Then $\exists$ a pos lin. functional $I : C_c(G) \to \mathbb{C}$, I is left inv, $I \neq 0$.

$$C_c(G) = \{ \text{cont. } f : G \to \mathbb{C} \mid \text{supp } f \text{ is comp} \}$$

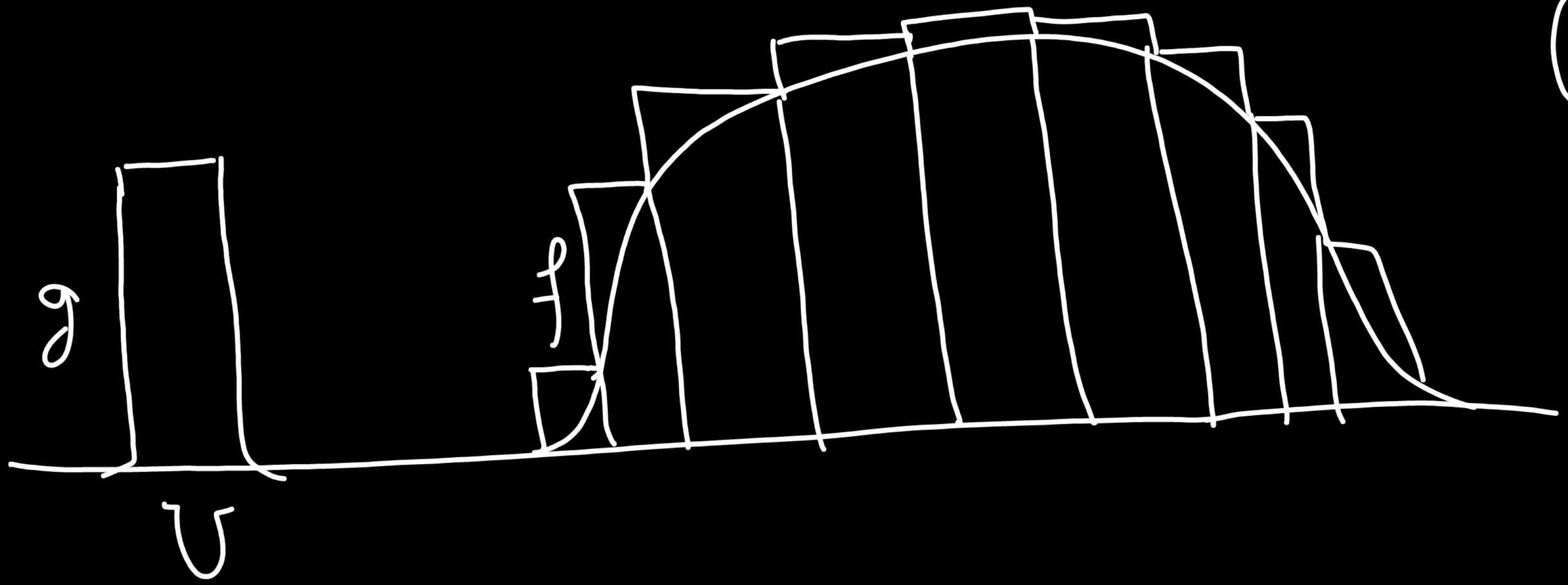
Lemma 1, $f, g \in C_c^+(G)$, $g \neq 0$
$\exists C > 0$ $\exists x_1, \dots, x_n \in G$ s.t. $f \leq C \sum_{i=1}^{n} L_{x_i} g$.

Notation

$$(f:g) = \inf\left\{ \sum_{i=1}^{n} c_i \; : \; f \leq \sum_{i=1}^{n} c_i L_{x_i} g \text{ for some } x_1, \dots, x_n \in G,\ c_i \geq 0 \right\}$$

("a relative appr. integral" of f rel. to g)

$$(f:g) \approx \frac{\int f\,dx}{\int g\,dx}$$



"an appr. integral" of f

Lemma 2 (1) $(cf : g) = c(f : g) \quad \forall c > 0$

(2) $(f_1 + f_2 : g) \le (f_1 : g) + (f_2 : g)$

(3) $(L_x f : g) = (f : g) \quad \forall x \in G$

(4) $(f : g) \ge \dfrac{\|f\|_\infty}{\|g\|_\infty}$. (5) $(f : g) \le (f : h)(h : g)$ $(h, g \ne 0)$.

Proof: exer.

Notation Choose $f_0 \in C_c^+(G)$, $f_0 \ne 0$.

Define, $\forall \varphi \in C_c^+(G) \setminus \{0\}$, $I_\varphi : C_c^+(G) \to (0, +\infty)$,

$$\boxed{I_\varphi(f) = \frac{(f : \varphi)}{(f_0 : \varphi)}}$$

Lemma 3

(1) $I_\varphi(cf) = c\, I_\varphi(f) \quad \forall c \geq 0$

(2) $I_\varphi(f_1 + f_2) \leq I_\varphi(f_1) + I_\varphi(f_2)$

(3) $I_\varphi(L_x f) = I_\varphi(f) \quad \forall x \in G.$

(4) $\dfrac{1}{(f_0 : f)} \leq I_\varphi(f) \leq (f : f_0)$

(if $f \neq 0$)

$(0, +\infty)^P = \{ a_f : f \in P \}$

$\underbrace{\prod S_f}_{= S} = \{ a_f : f \in P,\ a_f \in S_f \}$

Proof of (4)

$\dfrac{(f:\varphi)}{(f_0:\varphi)} \leq (f : f_0)$ (see L2 (5))

$\dfrac{(f:\varphi)}{(f_0:\varphi)} \geq \dfrac{1}{(f_0:f)} \iff \dfrac{(f_0:\varphi)}{(f:\varphi)} \leq (f_0 : f)$ True by L2, (5). $\square$

Lemma 4. Let $f_1, f_2 \in C_c^+(G)$. Then $\forall \varepsilon > 0$ $\exists$ nbhd $U \ni e$ s.t. $\forall \varphi \in C_c^+(G) \setminus \{0\}$ with $\operatorname{supp} \varphi \subset U$, we have

$$I_\varphi(f_1) + I_\varphi(f_2) \le I_\varphi(f_1 + f_2) + \varepsilon.$$

Proof of Thm Let $P = C_c^+(G) \setminus \{0\}$.

$\forall \varphi \in P \quad I_\varphi \in (0, +\infty)^P$.

$\forall f \in P$ let $S_f = \left[\frac{1}{(f_0 : f)}, (f : f_0)\right]$ $\quad L3 \Longrightarrow I_\varphi \in \overbrace{\prod_{f \in P} S_f}^{S \text{ is compact}}$

$\forall$ nbhd $U \ni e$ let $K_U = \{ I_\varphi \mid \varphi \in P, \operatorname{supp}\varphi \subset U \} \subset S$.

$K_U \neq \emptyset$

$U \subset V \Rightarrow K_U \subset K_V$. Hence $K_{U_1} \cap \dots \cap K_{U_n} \supset K_{U_1 \cap \dots \cap U_n} \neq \emptyset$

Hence $\{K_U : U \ni e\}$ has the finite intersec property

$$\underset{\text{(compactness)}}{\Longrightarrow} \bigcap_{U \ni e} K_U \neq \emptyset. \qquad \text{Let } \boxed{I \in \bigcap_{U \ni e} K_U} \subset S.$$

$I : P \to (0, +\infty)$. Claim: I is pos. hmg, additive, and left invar.

$$\left.\begin{array}{l}\forall U\ni e\quad \forall \varepsilon>0\quad \forall f_1,\dots,f_n\in P\quad \exists \varphi\in P \text{ with } \operatorname{supp}\varphi\subset U\\ \text{s.t. } |I(f_j)-I_\varphi(f_j)|<\varepsilon\quad \forall j=1,\dots,n\end{array}\right\}(*)$$

$(*)$ & L3 $\Longrightarrow$ I is pos. hmg, subadd, left invar.

$(*)$ & L4 $\Longrightarrow$ I is additive, that is,

$$I(f_1+f_2)=I(f_1)+I(f_2)\quad \forall f_1,f_2\in P.$$

Let $I(0)=0$.

$\forall f\in C_c(G)$ $\quad f=(f_1-f_2)+i(f_3-f_4)$, where $f_k\in C_c^+(G)$ $(k=1,\dots,4)$

Let $I(f)=I(f_1)-I(f_2)+i\left(I(f_3)-I(f_4)\right)$

Exer. $I : C_c(G) \to \mathbb{C}$ is well defined, linear, $I \neq 0$, and left invar. $\square$

Proof of L4. Let $f = f_1 + f_2 + \delta u$, where $\delta > 0$, and $u \in C_c^+(G)$ s.t. $u(x) = 1$ $\forall x \in \operatorname{supp}(f_1 + f_2)$.

$f_k = f h_k$ $(k = 1, 2)$ where $h_k \in C_c^+(G)$ (exer).

Suppose $f \leq \sum_{i=1}^{n} c_i L_{x_i} \varphi$ $\quad (c_i \geq 0,\ x_i \in G)$

Then $f_k \leq \sum c_i h_k L_{x_i} \varphi$ $(k=1,2)$, that is,

$$f_k(x) \leq \sum c_i h_k(x) \varphi(x_i^{-1}x) \qquad (*)$$

$\exists$ a nbhd $U \ni e$ s.t. $\forall x, y \in G$ satisfying $x^{-1}y \in U$ we have $|h_k(x) - h_k(y)| < \delta$. $(k=1,2)$

$$\boxed{I_\varphi(f) \leq (f : f_0)}$$

Suppose $\operatorname{supp}\varphi \subset U$.

If $x_i^{-1}x \notin U$, then the RHS of $(*)$ is 0.

If $x_i^{-1}x \in U$, then $|h_k(x) - h_k(x_i)| < \delta$

$$\Longrightarrow f_k(x) \leq \sum c_i (h_k(x_i) + \delta) \varphi(x_i^{-1}x) \quad \forall x \in G \qquad (k=1,2)$$

$$\Rightarrow (f_k : \varphi) \leq \sum c_i \big(h_k(x_i) + \delta\big) \qquad (k=1,2)$$

$$\Rightarrow (f_1 : \varphi) + (f_2 : \varphi) \leq \sum c_i (1+\delta) \qquad (\text{because } h_1 + h_2 \leq 1.)$$

$$\Rightarrow (f_1 : \varphi) + (f_2 : \varphi) \leq (1+\delta)(f : \varphi) \qquad \Big/ \, (f_0 : \varphi)$$

$$\Rightarrow I_\varphi(f_1) + I_\varphi(f_2) \leq (1+\delta) I_\varphi(f) \leq$$

$$\leq (1+\delta)\big(I_\varphi(f_1 + f_2 + \delta u)\big) \leq (1+\delta)\big(I_\varphi(f_1+f_2) + \delta I_\varphi(u)\big)$$

$$\leq I_\varphi(f_1+f_2) + \underbrace{\delta (f_1 + f_2 : f_0) + \delta(1+\delta)(u : f_0)}_{< \varepsilon \text{ if } \delta \text{ is small enough.}} \qquad \square$$

The uniqueness of the Haar measure.

Lemma 1 G = loc comp. group, μ = a Haar meas on G. Then

(1) $\forall\ \emptyset \neq U \subset G$ open we have $\mu(U) > 0$.

(2) If $f \in C(G)$ is μ-integrable, $f \geq 0$, $\int f\, d\mu = 0 \Rightarrow f = 0$.

Proof (1) Suppose $\mu(U) = 0$.

$\forall$ compact set $K \subset G$ $\quad K \subset x_1 U \cup \ldots \cup x_n U$ for some $x_1, \ldots, x_n$

$\Rightarrow \mu(K) = 0 \Rightarrow \mu = 0$ on open sets (by inner regularity)

$\Rightarrow \mu = 0$ on all Borel sets (by outer reg), a contr

(2) $f=0$ μ-a.e., that is, $\mu\big(\underbrace{f^{-1}((0,+\infty))}_{\text{open}}\big)=0 \xRightarrow[(1)]{} f=0.$ $\square$

Lemma 2. G = loc. comp. group, μ = a Rad meas on G.
Let $f\in C_c(G)$; define $g(x)=I_\mu(R_x f)$, $h(x)=I_\mu(L_x f)$.
Then g,h are cont.

Proof. (continuity of g at e).

$$|g(x)-g(e)|\le \int_G |f(yx)-f(y)|\,d\mu(y)$$

$\forall \varepsilon > 0 \ \exists$ nbhd $U \ni e$ s.t. $|f(yx) - f(y)| < \varepsilon \quad \forall y \in G, \forall x \in U$

Let $F = \operatorname{supp} f$,

choose a rel. compact, symm. nbhd $V \ni e$; let $K = F \cdot \overline{V}$.

K is compact.

Claim: if $y \notin K$, then $\boxed{f(y) = f(yx) = 0}$ $\quad (x \in V)$.

Indeed, if $f(yx) \neq 0$, then $yx \in F \Rightarrow y \in F \cdot x^{-1} \subset F \cdot V \subset K$

(a contr)

$$\Rightarrow |g(x) - g(e)| \le \int_K |f(yx) - f(y)| \, d\mu(y) < \varepsilon \mu(K) \Rightarrow g \text{ is cont at } e.$$

$(x \in V \cap U)$ $\quad$ <u>Exer</u>. Complete the proof $\square$

Thm 2. G = loc. comp group, μ, ν = (left) Haar measures on G.
$\Longrightarrow \exists c > 0$ s.t. $\nu = c\mu$.

Proof $\forall f \in C_c(G) \setminus \operatorname{Ker} I_\mu$ define $D_f : G \to \mathbb{C}$,

$$D_f(x) = \frac{I_\nu(R_x f)}{I_\mu f}.$$

L2 $\Rightarrow D_f$ is cont.

Claim D_f does not depend on f. $(*)$

If $(*)$ is true, then $I_\nu(f) = D(e)\, I_\mu(f)$ $\forall f \notin \operatorname{Ker} I_\mu$

$$\xrightarrow[\text{(exer)}]{} I_\nu = D(e) I_\mu \text{ everywhere on } C_c(G) \Longrightarrow \nu = c\mu, \text{ where } c = D(e).$$

<u>Let's prove (*)</u>

$$I_\mu(f) I_\nu(g) = \iint f(x) g(y)\, d\nu(y)\, d\mu(x) = \iint f(x) g(x^{-1}y)\, d\nu(y)\, d\mu(x)$$

$$\overset{\text{(Fub)}}{=} \iint f(x) g(x^{-1}y)\, d\mu(x)\, d\nu(y) = \iint f(yx) g(x^{-1})\, d\mu(x)\, d\nu(y)$$

$$\overset{\text{(Fub)}}{=} \iint f(yx) g(x^{-1})\, d\nu(y)\, d\mu(x) = \int I_\nu(R_x f) g(x^{-1})\, d\mu(x)$$

$$\Longrightarrow I_\nu(g) = \int D_f(x) g(x^{-1})\, d\mu(x)$$

Suppose; $f, f' \in C_c(G) \setminus \operatorname{Ker} I_\mu$.

$$\Longrightarrow \int (D_f - D_{f'}) g \, d\mu = 0 \quad \forall g \in C_c(G)$$

Replace g by $\overline{(D_f - D_{f'})} |g|^2 \Longrightarrow \int |(D_f - D_{f'}) g|^2 d\mu = 0$

$$\xRightarrow[L1]{} (D_f - D_{f'}) g = 0 \quad \forall g \in C_c(G) \Longrightarrow D_f = D_{f'}. \Longrightarrow (*).$$

$\square$