G = top group

<u>Def</u> $f: G \to \mathbb{C}$ is left (resp. right) <u>uniformly cont</u> if $\forall \varepsilon > 0$ $\exists$ nbhd $U \ni e$ st. $\forall x \in G$ $\forall u \in U$ $|f(x) - f(xu)| < \varepsilon$ (resp $|f(x) - f(ux)| < \varepsilon$)

<u>Equiv</u>: f is left (resp. right) unif cont if $\forall \varepsilon > 0$ $\exists$ nbhd $U \ni e$ s.t $|f(x) - f(y)| < \varepsilon$ whenever $x^{-1}y \in U$ (resp. $yx^{-1} \in U$)

Prop. G = loc. comp. group, $f \in C_c(G)$. Then f is left and right unif. cont.

Lemma. X, Y, Z top. spaces; $F : X \times Y \longrightarrow Z$ cont.; $Z_0 \subset Z$ open, $Y_0 \subset Y$ compact.
Let $X_0 = \{x \in X : F(x,y) \in Z_0 \; \forall y \in Y_0\}$. Then X_0 is open

Proof : exer.

Proof of Prop. f is left unif cont $\Longleftrightarrow$ Sf is right unif cont. $\left((Sf)(x) = f(x^{-1}) \right)$

Let's show that f is left unif cont

Let $F = \operatorname{supp} f$; $V \ni e$ a rel. compact symm. nbhd of e $(V^{-1} = V)$

Let $K = F \cdot \overline{V}$. K is compact. Let $\varepsilon > 0$;

let $W = \{ y \in G : \forall x \in K \ |f(x) - f(xy)| < \varepsilon \}$.

Lemma $\Rightarrow$ W is open; $e \in W$. Let $U = V \cap W$.

If $x \in K$, $y \in U \Rightarrow |f(x) - f(xy)| < \varepsilon$.

Suppose $x \in G \setminus K$, $y \in U$. Then $f(x) = 0$. Claim: $f(xy) = 0$

If not, then $xy \in F \Rightarrow x \in F \cdot y^{-1} \subset F \cdot V \subset K$, a contr. $\_$.

$\Rightarrow |f(x)-f(xy)|<\varepsilon \ \forall x\in G, \forall y\in U.$ $\square$

The Haar measure.

G = loc. comp. group, μ = a Rad. meas on G. (positive)

Def μ is left (resp right) invariant if $\forall x\in G$ $\forall$ Borel $B\subset G$

$\mu(xB)=\mu(B)$ (resp $\mu(Bx)=\mu(B)$)

If, moreover, $\mu\neq 0$, then μ is a left (resp. right) Haar measure

Observe. If μ is left inv, then $\nu(B)=\mu(B^{-1})$ is right invar.

$$\{\text{left inv}\} \rightleftarrows \{\text{right inv}\}.$$

Convention. Haar measure = left Haar measure

Examples. (1) The counting meas on a discr. group.

(2) The Lebesgue meas on $\mathbb{R}^n$.

(3) The $\frac{\text{length meas on } \mathbb{T}}{2\pi}$

<u>Thm</u>: (A. Haar, J. von Neumann, A. Weil).

G = loc. comp. group

(1) $\exists$ a Haar meas on G

(2) If μ, ν are Haar measures $\Longrightarrow \exists\, c > 0$ s.th $\nu = c\mu$.

<u>Haar meas on Lie groups.</u>

G = real Lie group, $n = \dim G$.

Choose $\omega_e \in \Lambda^n(T_e^* G)$, $\omega_e \neq 0$.

$\forall x \in G \quad l_x : G \to G, \quad l_x(y) = xy$

$dl_{x^{-1}} = l_{x^{-1},*} : T_x G \to T_e G \qquad l^*_{x^{-1}} : \Lambda(T_e^* G) \to \Lambda(T_x^* G)$

Let $\omega_x = l^*_{x^{-1}} \omega_e \in \Lambda^n(T_x^* G)$. $\qquad \omega \in \Omega^n(G)$. $\omega_x \neq 0$.

In particular, G is orientable

Choose an orientation on G such that ω is positive.

$\forall$ Borel $B \subset G$ define $\mu(B) = \int_B \omega$.

<u>Claim</u>: μ is a Haar meas.

Indeed: $\quad l_x^* \omega = \omega \ \forall x$ by constr.

μ is a Radon meas (because $\mu(K)<\infty\ \forall$ compact $K\subset G$)
and G is 2nd countable

$$\mu(xB)=\int_{xB}\omega = \int_B l_x^*\omega \qquad (l_x \text{ is orientation-preserving})$$

$$=\int_B \omega = \mu(B) \implies \mu \text{ is left inv.}$$

<u>Coordinate form of ω</u>

$x^1,\dots,x^n$ = coordinates in a nbhd of p

$y^1,\dots,y^n$ coordinates in a nbhd of e; $\omega_e = dy^1\wedge\dots\wedge dy^n$.

$$\forall p\in G \qquad \omega(p)=\det\left(l_{p^{-1},*}(p)\right)dx^1\wedge\dots\wedge dx^n =$$

$$= \det\left(\frac{\partial (y^i \circ \ell_{p^{-1}})}{\partial x^j}(p) \right) dx^1 \wedge \dots \wedge dx^n.$$

<u>Example 1</u> $G = \mathbb{R}^\times = (\mathbb{R}\setminus\{0\}, \cdot)$ $\quad p \in G \quad \omega(p) = \frac{dx}{p}.$

x = the glob coord on $\mathbb{R}$

($x = id_{\mathbb{R}}$)

The orient. of $\mathbb{R}^\times$ compatible with ω is

- the standard orient on $\mathbb{R}_{>0}$
- the opposite orient on $\mathbb{R}_{<0}$

(λ = Leb meas)

$$\forall f \in C_c(\mathbb{R}^\times) \quad \int_{\mathbb{R}^\times} f \, d\mu = \int_{\mathbb{R}^\times} \frac{f(x)}{|x|} dx, \text{ i.e., } \boxed{\mu = \frac{\lambda}{|x|}.}$$

Example/exer 2 $G = GL_n(\mathbb{R})$

Prove: $\mu_{left} = \mu_{right} = \frac{\lambda}{|\det|^n}$.

Example/exer 3.

$\forall a, b \in \mathbb{R}$ $(a \neq 0)$ $L_{a,b} : \mathbb{R} \to \mathbb{R}$, $L_{a,b}(x) = ax + b$.

$$G = \{L_{a,b} : a \in \mathbb{R}^\times, b \in \mathbb{R}\} = \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}^\times, b \in \mathbb{R}\right\},$$

Find explicitly (in terms of a, b)

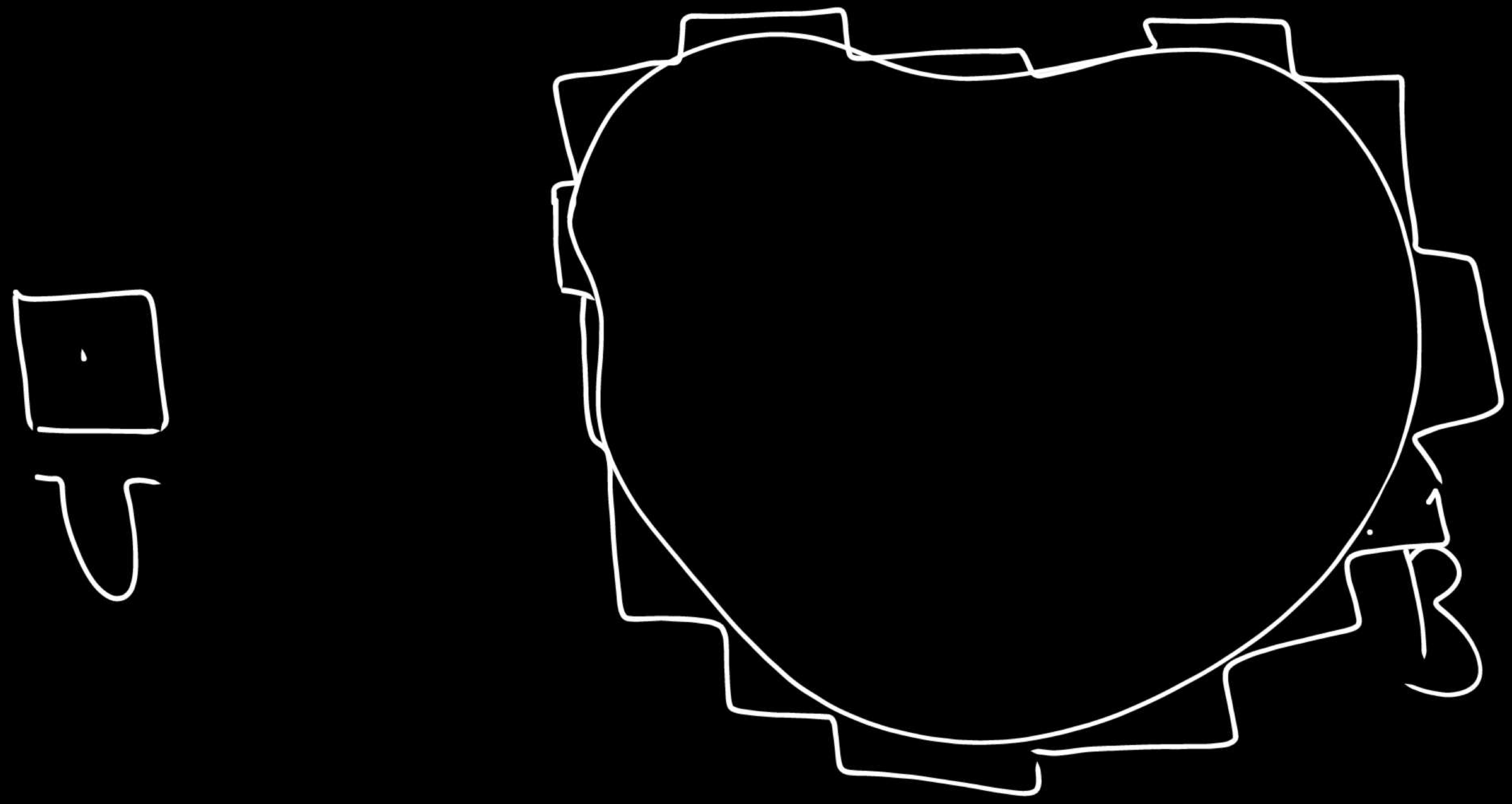
μ_{left} and μ_{right}; show that $\mu_{left} \neq \mu_{right}$.

The existence of a Haar measure

G = loc comp. group.

A rough idea: $U \subset G$ nbhd of e.

$\forall$ Borel $B \subset G$ let $(B:U) = \min\{n : B \subset x_1 U \cup \ldots \cup x_n U$ for some $x_1, \ldots, x_n \in G\}$.



$$(B:U) \approx \frac{\text{"area of } B\text{"}}{\text{area of } U}$$

$$U \to \{e\} \Rightarrow (B:U) \to \infty.$$

Choose $K \subset G$ compact, $\operatorname{Int} K \neq \emptyset$.

$$\lim_{U \to \{e\}} \frac{(B:U)}{(K:U)} = \mu(B).$$

<u>Notation.</u> 1) μ = a Rad meas on G, $x \in G$.

Define Rad. meas $L_x\mu$, $R_x\mu$:

$$(L_x\mu)(B) = \mu(x^{-1}B), \quad (R_x\mu)(B) = \mu(Bx)$$

We have $\boxed{L_{xy} = L_x L_y \, ; \quad R_{xy} = R_x R_y}$ $(*)$

μ is left inv $\Longleftrightarrow L_x\mu = \mu \ \forall x$

2) $f \in \mathrm{Fun}(G) = \mathbb{C}^G$.

$L_x f, R_x f \in \mathrm{Fun}(G)$ $\quad (L_x f)(y) = f(x^{-1}y), \quad (R_x f)(y) = f(yx)$

(*) holds.

3) Let $I : C_c(G) \to \mathbb{C}$ be a lin functional.

Define functionals $L_x I, R_x I : C_c(G) \to \mathbb{C}$

$$L_x I = I \circ L_{x^{-1}}, \quad R_x I = I \circ R_{x^{-1}}.$$

(*) holds. (exer.)

Prop μ = a Rad meas on G, $x \in G$, $I_\mu(f) = \int f\,d\mu$ $(f \in C_c(G))$.

Then $L_x I_\mu = I_{L_x \mu}$.

Proof $(L_x I_\mu)(\chi_B) = (I_{L_x \mu})(\chi_B)$ $\forall$ Borel $B \subset G$. (exer)

$\Rightarrow (L_x I_\mu)(f) = (I_{L_x \mu})(f)$ $\forall$ bdd Borel fun. f concentrated on a set of fin. meas

$\square$

Cor. μ is left inv $\iff$ I_μ is left inv.

Thm! G = loc. comp. group

(1) $\exists$ a left inv. positive lin functional I on $C_c(G)$, $I \neq 0$

(2) If I, J are such functionals, then $\exists\, c > 0$ s.t. $J = cI$.

Lemma 1 $f, g \in C_c^+(G)$, $g \neq 0$.

$\Rightarrow \exists\, C > 0\ \exists\, x_1, \dots, x_n \in G$ s.t. $f \leq C \sum_{i=1}^{n} L_{x_i} g$.

Proof $\exists\, \varepsilon > 0\ \exists$ open $U \subset G$, $U \neq \emptyset$, s.t. $g(x) > \varepsilon\ \forall x \in U$.

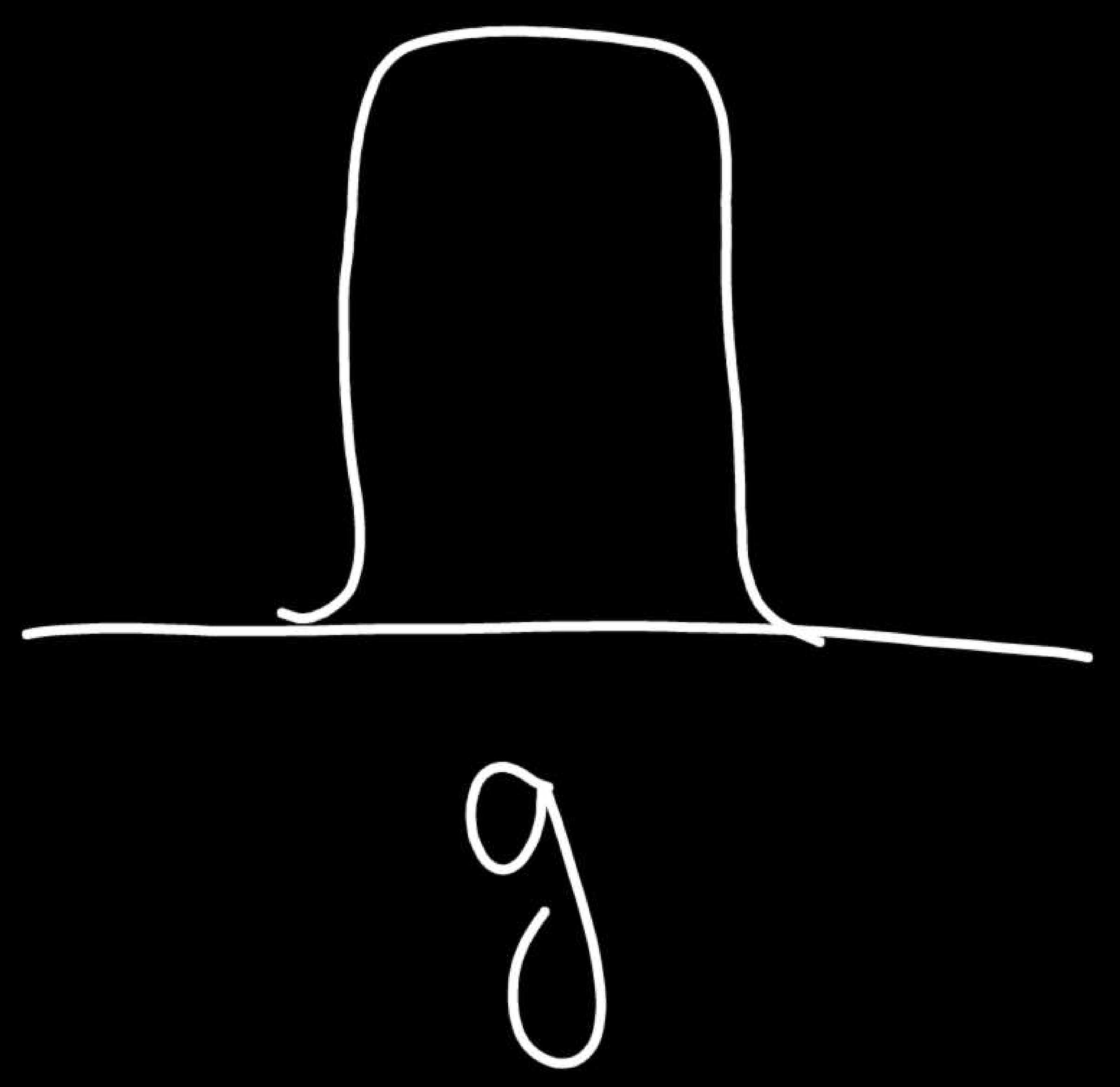
$\operatorname{supp}(f) \subset \bigcup_{i=1}^{n} x_i U$ for some $x_1, \dots, x_n \in G$ $\square$

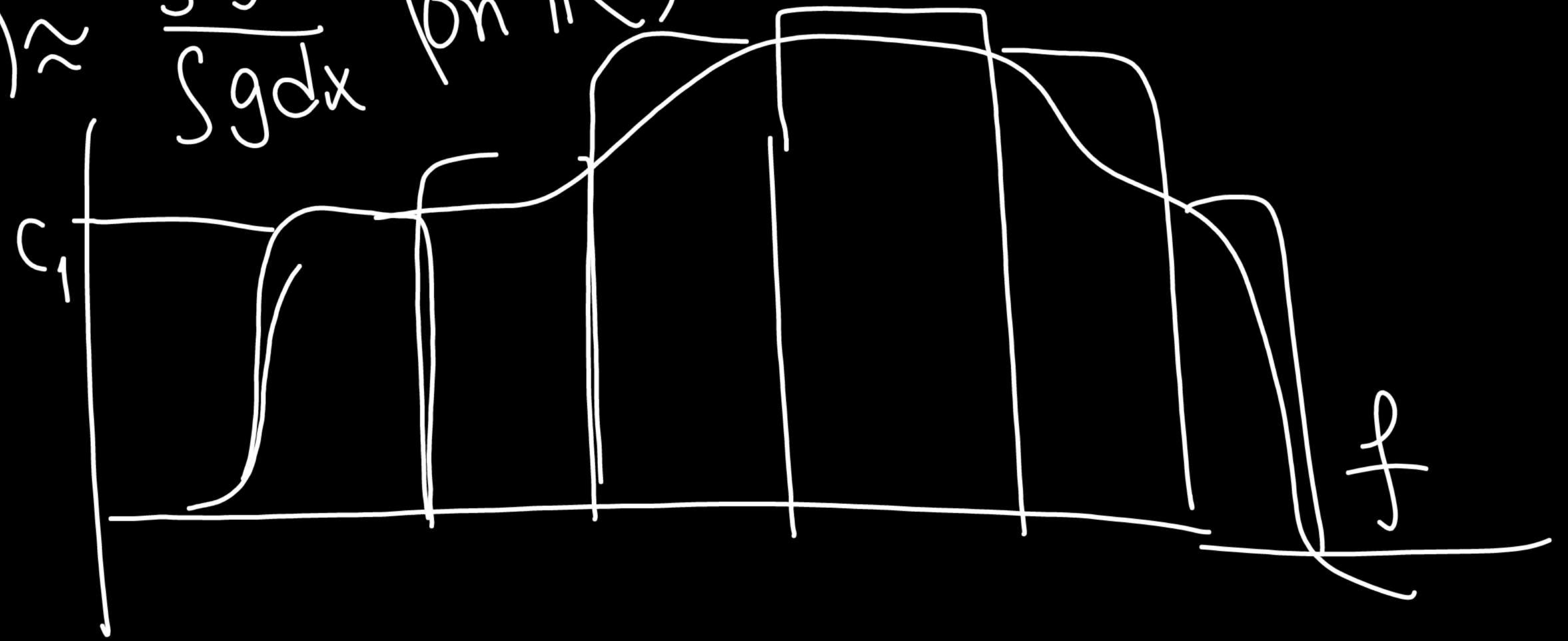
Notation. $f, g \in C_c^+(G)$, $g \neq 0$

$$(f:g) = \inf\left\{ \sum_{i=1}^n c_i : f \le \sum_{i=1}^n c_i L_{x_i} g \text{ for some } x_1, \dots, x_n \in G \right\}$$

"relative integral".

Geom. idea: $(f:g) \approx \dfrac{\int f\,dx}{\int g\,dx}$ (on $\mathbb{R}$)





Lemma 2. (1) $(cf:g) = c(f:g)$ $\forall c>0$

(2) $(f_1+f_2:g) \le (f_1:g) + (f_2:g)$

(3) $(L_x f:g) = (f:g)$ $\forall x$;

(4) $(f:g) \ge \dfrac{\|f\|_\infty}{\|g\|_\infty}$. $(f, g \ne 0)$

(5) $(f:g) \le (f:h)(h:g)$ $(h, g \ne 0)$

Proof: exer.

Rem (4) $\Rightarrow (f:g) > 0$ if $f, g \ne 0$.