$f \in L^1(\mathbb{R})$ $\qquad \hat{f}: \mathbb{R} \to \mathbb{C}$ $\qquad \hat{f}(\lambda) = \int_{\mathbb{R}} f(t) e^{-2\pi i \lambda t}\, dt.$

$\hat{f} \in C_0(\mathbb{R})$

$\mathcal{F}: L^1(\mathbb{R}) \to C_0(\mathbb{R}), \quad f \mapsto \hat{f}.$ $\qquad \|\hat{f}\|_\infty \leq \|f\|_1.$

$(f * g)^\wedge = \hat{f}\,\hat{g}.$

$\mathcal{F}(L^1(\mathbb{R})) = A(\mathbb{R}) \subsetneq C_0(\mathbb{R})$ the <u>Fourier alg.</u>

Thm. (1) (uniqueness thm)

$\mathcal{F} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$ is inj.

(2) (density thm)

$A(\mathbb{R})$ is dense in $C_0(\mathbb{R})$

(3) (Plancherel thm)

$\mathcal{F}((L^1 \cap L^2)(\mathbb{R})) \subset L^2(\mathbb{R})$, and $\mathcal{F}|_{L^1 \cap L^2}$ uniquely extends to a unitary isom $\mathcal{F}^\circ : L^2(\mathbb{R}) \xrightarrow{\sim} L^2(\mathbb{R})$

Moreover, $(\mathcal{F}^\circ)^2 = S$. $\quad (f \mapsto (t \mapsto f(-t)))$

(4) (inversion formula)

Let $f \in L^1(\mathbb{R})$. Then:

$$\hat{f} \in L^1(\mathbb{R}) \iff \exists f_0 \in A(\mathbb{R}) \text{ s.t. } f \underset{a.e.}{=} f_0.$$

If they hold, then $f_0 = S\hat{\hat{f}}$. That is,

$$f(t) \underset{ae}{=} f_0(t) = \int \hat{f}(\lambda) e^{2\pi i \lambda t} d\lambda.$$

Ingredients of the proof

Lemma. (1) $f \in C^1(\mathbb{R})$, $f, f' \in L^1(\mathbb{R}) \implies \hat{f'}(\lambda) = 2\pi i \lambda \hat{f}(\lambda)$.

(2) $f \in C^p(\mathbb{R})$, $f, \dots, f^{(p)} \in L^1(\mathbb{R}) \implies \hat{f}(\lambda) = o(|\lambda|^{-p})$ $(\lambda \to \infty)$

(3) $f, tf \in L^1(\mathbb{R})$ (where $t = \mathrm{id}_{\mathbb{R}}$) $\Longrightarrow \hat{f} \in C^1(\mathbb{R})$, and

$$\hat{f}'(\lambda) = -2\pi i \, (tf)^{\wedge}(\lambda).$$

(4) $f, tf, \ldots, t^p f \in L^1(\mathbb{R}) \Longrightarrow \hat{f} \in C^p(\mathbb{R})$.

<u>Def</u> The <u>Schwartz space</u> is

$$\mathcal{S}(\mathbb{R}) = \{ f \in C^\infty(\mathbb{R}) : \forall k, l \in \mathbb{Z}_{\geq 0} \ \ t^k f^{(l)} \text{ is bdd} \}.$$

$$\|f\|_{k,l} = \sup_{t \in \mathbb{R}} |t^k f^{(l)}(t)|.$$

$$\{ \|\cdot\|_{k,l} : k, l \in \mathbb{Z}_{\geq 0} \}$$

$$\downarrow$$

topology on $\mathcal{S}(\mathbb{R})$

Thm. $\mathcal{F}(\mathcal{S}(\mathbb{R}))=\mathcal{S}(\mathbb{R})$, and
$\mathcal{F}:\mathcal{S}(\mathbb{R})\to\mathcal{S}(\mathbb{R})$ is a topol. isomorphism.
Moreover, $\mathcal{F}^2=S$ on $\mathcal{S}(\mathbb{R})$.

Lemma/exer 0.
$E, F=$ vec spaces; $P=\{|\cdot|_i\ i\in I\}$, $Q=\{\|\cdot\|_j : j\in J\}$
families of seminorms on E, F resp
$T:E\to F$ linear. Then T is cont $\iff$

$$\forall j\in J\ \exists C>0\ \exists i_1,\dots,i_n\in I \text{ s.t. } \forall v\in E\quad \|Tv\|_j\le C\max_{1\le l\le n}\|v\|_{i_k}$$

Lemma/exer 1 Let $\hat{\mathcal{F}} = S\mathcal{F} = \mathcal{F}S$

Then $\mathcal{F}(\mathcal{S}(\mathbb{R})) \subset \mathcal{S}(\mathbb{R})$, and $\mathcal{F}, \hat{\mathcal{F}} : \mathcal{S} \to \mathcal{S}$ are cont.

Lemma/exer 2 Define $M, D : \mathcal{S} \to \mathcal{S}$, $Mf = tf$,

$D = \frac{1}{2\pi i}\frac{d}{dt}$. Then $\mathcal{F}D = M\mathcal{F}$, $\mathcal{F}M = -D\mathcal{F}$.

Lemma/exer 3. Let $T = \hat{\mathcal{F}}\mathcal{F}$. Then $T = \mathcal{F}\hat{\mathcal{F}}$,

and $TM = MT$, $TD = DT$.

Lemma/exer 4. Suppose $T : \mathcal{S} \to \mathcal{S}$ is a lin map

s.t. $TM = MT$, $TD = DT \implies T = c\mathbb{1}$ for some $c \in \mathbb{C}$.

Hint. $\forall a \in \mathbb{R} \quad m_a = \{ f \in \mathcal{S}(\mathbb{R}) : f(a) = 0 \}$.

$TM = MT \Longrightarrow T(m_a) \subset m_a \ \forall a \Longrightarrow \exists\, c \in C^\infty(\mathbb{R})$ s.t.

$$Tf = cf \quad \forall f \in \mathcal{S}.$$

$TD = DT \Longrightarrow c = \text{const}.$

Lemma / exer 5. $f(t) = e^{-\pi t^2}$. $\longrightarrow \hat{f} = f$.

Hint. $f' + 2\pi t f = 0 \Longrightarrow \hat{f}' + 2\pi t \hat{f} = 0 \Longrightarrow$

$\Longrightarrow \hat{f} = cf \quad (c \in \mathbb{C})$; $f(0) = 1 = \hat{f}(0) = \int e^{-\pi t^2} dt$

$\Longrightarrow c = 1.$

Notation

$\mathcal{S}'(\mathbb{R})$ = the top. dual of $\mathcal{S}$ = $\{$cont. lin. functionals $\mathcal{S}(\mathbb{R}) \to \mathbb{C}\}$.

(The space of tempered distributions)

Exer. $p \in [1, +\infty]$

$$L^p(\mathbb{R}) \hookrightarrow \mathcal{S}'(\mathbb{R}), \quad f \mapsto \left(\varphi \mapsto \int f \varphi \, dt\right).$$

__Notation.__

$\mathcal{F}' : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$, $\quad \mathcal{F}'g = g \circ \mathcal{F}$.

(that is, $\mathcal{F}'$ is dual to $\mathcal{F} : \mathcal{S} \to \mathcal{S}$).

$\mathcal{F}'$ is the __Fourier transform on $\mathcal{S}'$.__

$\mathcal{F}' : \mathcal{S}' \to \mathcal{S}'$ is an isom.

__Exer.__ (1)

$$\begin{tikzcd} L^1 \arrow[r, "\mathcal{F}"] \arrow[d, hook] & C_0 \arrow[d, hook] \\ \mathcal{S}' \arrow[r, "\mathcal{F}'"] & \mathcal{S}' \end{tikzcd}$$

comm. $\Longrightarrow$ uniqueness thm.

(2)
$$\begin{tikzcd} \mathcal{S} \arrow[r, "\mathcal{F}"] \arrow[d, hook] & \mathcal{S} \arrow[d, hook] \\ L^1 \arrow[r, "\mathcal{F}"] & C_0 \end{tikzcd}$$

$\mathcal{S}$ is dense in C_0

$\Longrightarrow \mathcal{F}(L^1)$ is dense in C_0

(3)
$$\begin{tikzcd} \mathcal{S} \arrow[r, "\mathcal{F}", "\text{unitary}"'] \arrow[d, hook] & \mathcal{S} \arrow[d, hook] \\ L^2 \arrow[r, "\mathcal{F}^{\cdot}", "\text{unitary}"'] \arrow[d, hook] & L^2 \arrow[d, hook] \\ \mathcal{S}' \arrow[r, "\mathcal{F}'"'] & \mathcal{S}' \end{tikzcd}$$

$\mathcal{F}'$ extends both $\mathcal{F}^{\cdot} : L^2 \to L^2$

$\mathcal{F}_{L^1} : L^1 \to C_0$

$\Longrightarrow \mathcal{F}^{\cdot}\big|_{L^1 \cap L^2} = \mathcal{F}\big|_{L^1 \cap L^2}$

$\Longrightarrow$ Plancherel.

Locally compact spaces. Radon measures

Def. A top space X is locally compact if $\forall x \in X$ $\exists$ a nbhd $U \ni x$ s.t. $\overline{U}$ is compact.

Examples (1) compact (2) discrete (3) $\mathbb{R}^n$ (4) any C^0-manifold.

Nonexamples. (1) $\mathbb{Q}$ (2) an inf-dim. normed space (3) (exer) an inf product of noncompact spaces

<u>Exer.</u> The prod of finitely many LC spaces is LC.

<u>Thm</u> (Urysohn's lemma)

X = a loc. comp. Hausd. top space, $K, F \subset X$, $K \cap F = \emptyset$

K is comp, F is closed $\Longrightarrow \exists$ a cont. $\varphi : X \to [0,1]$

s.t. $\varphi|_K = 1$, $\varphi|_F = 0$, $\operatorname{supp} \varphi$ is comp.

X = a Hausdorff loc. comp. top. space.

<u>Notation.</u> $\mathcal{B}or(X)$ = Borel σ-algebra on X =

= the smallest σ-subalg of 2^X containing open sets.

<u>Def.</u> A (positive) <u>Borel meas</u> on X is a σ-add measure

$\mu : \mathrm{Bor}(X) \to [0, +\infty]$

<u>Def</u> μ = a Bor. meas on X, $B \subset X$ Borel. μ is

(1) <u>outer regular</u> on B if $\mu(B) = \inf\left\{\mu(U) : \begin{array}{l} U \supset B \\ U \text{ open} \end{array}\right\}$.

(2) <u>inner regular</u> on B if $\mu(B) = \sup\{\mu(K) : K \subset B \text{ compact}\}$

(an <u>outer</u> Radon meas)

<u>Def.</u> μ is a <u>Radon meas</u> if

(1) $\forall$ comp. $K \subset X$, $\mu(K) < \infty$.

(2) μ is outer regular on all Borel sets

(3) μ is inner regular on all open sets.

<u>Example</u> (1) The Lebesgue meas on $\mathbb{R}^n$.

(2) X is discrete. $\mu(A) = \begin{cases} \text{Card } A & \text{if } A \text{ is fin} \\ +\infty & \text{if } A \text{ is inf} \end{cases}$ (counting meas)

Facts / exer.

(1) Suppose X is σ-comp (i.e., $X = \bigcup_{n \in \mathbb{N}} X_n$, X_n is comp).
Then each Radon meas on X is inner reg on all Borel sets.

(2) Suppose X is 2nd countable, μ = a Borel meas on X s.t. $\mu(K) < \infty$ $\forall$ comp $K \subset X$.
Then μ is inner reg and outer reg on all Borel sets.

Notation

$$C_c(X) = \{ f \in C(X) : \operatorname{supp} f \text{ is compact} \}.$$

$f \in C_c(X)$ $\quad f \geq 0 \overset{}{\underset{\text{def}}{\iff}} \forall x \in X \; f(x) \geq 0.$

$f, g \in C_c(X)$ $\quad f \leq g \underset{\text{def}}{\iff} g - f \geq 0.$

$$C_c^+(X) = \{ f \in C_c(X) : f \geq 0 \}.$$

Def A lin. functional $I : C_c(X) \to \mathbb{C}$ is <u>positive</u> $(I \geq 0)$ if $I(f) \geq 0 \;\; \forall f \geq 0.$

Example. μ = a pos. Radon meas on X.

$$I_\mu : C_c(X) \to \mathbb{C}, \quad I_\mu(f) = \int_X f \, d\mu \implies I_\mu \geq 0.$$

Thm. (Riesz, Markov, Kakutani)

$\exists$ a bijection

$$\left\{ \begin{array}{c} \text{Pos. Radon} \\ \text{measures on } X \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{Pos. linear functionals} \\ \text{on } C_c(X) \end{array} \right\}$$

$$\mu \longmapsto I_\mu$$

Locally compact groups.

Def. A topological group is a group G equipped with a topol. such that

$$\left.\begin{array}{l} G\times G \to G, \ (x,y)\mapsto xy \\ G\to G, \ x\mapsto x^{-1} \end{array}\right\} \text{ are cont.}$$

Observe: (1) $\forall x\in G$ the maps $y\mapsto xy$ and $y\mapsto yx$ are homeomorphism $G\to G$.

(2) $x\mapsto x^{-1}$ is a homeo $G\to G$.

Notation $S, T \subset G$

$$ST = \{xy : x \in S, y \in T\}. \qquad S^{-1} = \{x^{-1} : x \in S\}.$$

S is **symmetric** if $S = S^{-1}$.

Observe: every nbhd $U \ni e$ contains a symm. nbhd of e (namely $U \cap U^{-1}$)

Def. A **locally comp group** is a loc. comp Hausdorff top. group

<u>Examples.</u> (1) discrete groups

(2) $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^\times$, $\mathbb{C}$, $\mathbb{C}^\times$, $\mathbb{T}$, $\mathbb{Q}_p$, $\mathbb{Q}_p^\times$, $\mathbb{Z}_p$

(3) $GL_n(\mathbb{K})$, $SL_n(\mathbb{K})$ $(\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\})$, U_n, SU_n,

O_n, SO_n, $\cdots$

(4) Any Lie group

<u>Def.</u> $G=$ top group, $f: G \to \mathbb{C}$.

f is <u>left</u> (resp <u>right</u>) <u>uniformly continuous</u> if $\forall \varepsilon > 0$ $\exists$ a nbhd $U \ni e$ s.t. $\forall x \in G$, $\forall u \in U$ we have $|f(x) - f(xu)| < \varepsilon$ (resp. $|f(x) - f(ux)| < \varepsilon$).

<u>Rem</u> For $G = \mathbb{R}$ we get the "usual" uniform continuity

Equivalently: f is left (resp. right) uniformly cont iff $\forall \varepsilon > 0$ $\exists$ a nbhd $U \ni e$ s.t. $\forall x, y \in G$ satisfying $x^{-1}y \in U$ (resp. $yx^{-1} \in U$) we have $|f(x) - f(y)| < \varepsilon$.