Nuclear spaces

(exercises for Lecture 17–20)

15.1. Let X be a nuclear locally convex space. Show that, for each Banach space E, every continuous linear map from X to E is binuclear.

15.2. Let X be a nuclear locally convex space. Is it true that, for each *normed* space E, every continuous linear map from X to E is nuclear?

15.3. Given a complete locally convex space X, show that the following conditions are equivalent:

(i) X is nuclear;
(ii) $X \cong \varprojlim(H_i, \varphi_{ij})$, where H_i are Hilbert spaces and the connecting maps $\varphi_{ij} \colon H_j \to H_i$ are nuclear for all $i < j$;
(iii) $X \cong \varprojlim(H_i, \varphi_{ij})$, where H_i are Hilbert spaces, the projective system (H_i, φ_{ij}) is reduced, and the connecting maps $\varphi_{ij} \colon H_j \to H_i$ are nuclear for all $i < j$;
(iv) same as (iii), but H_i are Banach spaces.

15.4. Let S^n be the n-sphere, and let $p \in S^n$. Construct a topological isomorphism

$$\mathscr{S}(\mathbb{R}^n) \cong \{ f \in C^\infty(S^n) : f \text{ is flat at } p \},$$

where "flat" means that all partial derivatives of f (w.r.t. any local coordinates) vanish at p. As a corollary (see the lecture), $\mathscr{S}(\mathbb{R}^n)$ is nuclear.

15.5. Show that the strongest locally convex space of uncountable dimension (or, equivalently, the locally convex direct sum of uncountably many copies of $\mathbb{K}$) is not nuclear.

15.6. Let X be a locally compact, Hausdorff, 1st countable (e.g. metrizable) topological space. Show that $C(X)$ is nuclear iff X is discrete.

15.7 (*the Grothendieck-Pietsch criterion)***.** Prove that a Köthe space $\lambda^1(I, P)$ is nuclear iff for every $p \in P$ there exist $q \in P$ and $\lambda \in \ell^1(I)_{\geqslant 0}$ such that $p_i \leqslant \lambda_i q_i$ for all $i \in I$. (We proved this at the lecture under the assumption that $I = \mathbb{N}$ and $p_i > 0$ for all $p \in P$, $i \in I$.)

15.8. Construct a non-nuclear Fréchet-Montel space. (*Hint:* try $\lambda^1(\mathbb{N}, P)$ for a suitable P.)

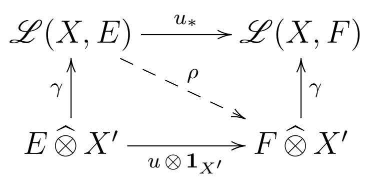
15.9. Let $0 \to X_1 \to X_2 \to X_3 \to 0$ be an exact sequence of Fréchet spaces, and let Y be a Fréchet space. Suppose that X_1 is nuclear. Show that the sequence

$$0 \to X_1 \mathbin{\widehat{\otimes}_\pi} Y \to X_2 \mathbin{\widehat{\otimes}_\pi} Y \to X_2 \mathbin{\widehat{\otimes}_\pi} Y \to 0$$

is exact. (This was proved at the lecture under the assumption that either Y or X_2 is nuclear.)

15.10. Let X be a complete nuclear locally convex space. Is X'_β (the strong dual of X) necessarily nuclear?

15.11. Let E and F be Banach spaces, and let $u \colon E \to F$ be a nuclear operator. Show that for each bornological locally convex space X there exists a continuous linear operator $\rho \colon \mathscr{L}(X, E) \to F \mathbin{\widehat{\otimes}} X'$ making the following diagram commute:



Here $\mathscr{L}(X,E)$ is equipped with the topology of uniform convergence on bounded subsets of X (and similarly for $\mathscr{L}(X,F)$ and for X'), u_* acts by the rule $\varphi \mapsto u \circ \varphi$, and γ is uniquely determined by $\gamma(y \otimes f)(x) = f(x)y$.

15.12. Let E and F be Banach spaces, and let $u\colon E \to F$ be a nuclear operator. Show that for each complete locally convex space Y there exists a continuous linear operator $\rho\colon \mathscr{L}(E',Y) \to Y \mathbin{\widehat{\otimes}} F$ making the following diagram commute:

$$\begin{array}{ccc} \mathscr{L}(E',Y) & \xrightarrow{\;u_*\;} & \mathscr{L}(F',Y) \\ {\scriptstyle\gamma}\uparrow & \overset{\rho}{\searrow} & \uparrow{\scriptstyle\gamma} \\ Y \mathbin{\widehat{\otimes}} E & \xrightarrow[\mathbf{1}_Y \otimes u]{} & Y \mathbin{\widehat{\otimes}} F \end{array}$$

Here $\mathscr{L}(E',Y)$ is equipped with the topology of uniform convergence on the unit ball of E' (and similarly for $\mathscr{L}(F',Y)$), u_* acts by the rule $\varphi \mapsto \varphi \circ u'$, and γ is uniquely determined by $\gamma(y \otimes x)(f) = f(x)y$.

15.13. Let X be a nuclear Fréchet space, and let E be a complete bornological locally convex space having a fundamental sequence $(B_n)_{n\in\mathbb{N}}$ of bounded sets (which means that each bounded subset of E is contained in some B_n). Show that each continuous linear operator from E to X is nuclear.

15.14. Let X be a complete locally convex space. Assume that, for each Banach space E, all continuous linear maps from E to X are nuclear. Show that X'_β (the strong dual of X) is nuclear.

Remark 15.1. Combining Exercises 15.13 and 15.14, we obtain an alternative proof of the fact that the strong dual of a nuclear Fréchet space is nuclear.