Nuclear operators

(EXERCISES FOR LECTURE 16)

14.1. Let X and Y be Banach spaces, and let $\varphi\colon X \to Y$ be a nuclear operator. Suppose also that Y_0 is a dense vector subspace of Y. Show that there exist sequences $y_n \to 0$ in Y_0, $f_n \to 0$ in X', and (λ_n) in $\mathbb{K}$ with $\sum_n |\lambda_n| < \infty$ such that $\varphi(x) = \sum_n \lambda_n f_n(x) y_n$ for all $x \in X$.

14.2. Let $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ be bounded linear operators between Banach spaces. Show that, if either φ or ψ is nuclear, then so is $\psi\varphi$.

14.3. Let $X = \ell^p$ (where $1 \leqslant p \leqslant \infty$) or $X = c_0$. Given $\alpha = (\alpha_n) \in \ell^\infty$, consider the diagonal operator

$$M_\alpha\colon X \to X, \qquad (x_1, x_2, \ldots) \mapsto (\alpha_1 x_1, \alpha_2 x_2, \ldots).$$

Show that M_α is nuclear iff $\alpha \in \ell^1$, and that $\|M_\alpha\|_{\mathcal{N}} = \|\alpha\|_1$.

(*Hint:* for $1 < p < \infty$, use Exercise 13.2 (a)).

14.4. Let $I = [a, b]$ and let $K \in C(I \times I)$. Prove that the integral operator

$$\varphi\colon C(I) \to C(I), \qquad (\varphi f)(x) = \int_a^b K(x, y) f(y)\, dy,$$

is nuclear.

14.5. Let φ be a bounded linear operator on ℓ^1 with matrix (a_{ij}) in the standard basis (e_j) (i.e., $\varphi e_j = \sum_i a_{ij} e_i$ for all $j \in \mathbb{N}$). Find a nuclearity criterion for φ and calculate the nuclear norm of φ in terms of the matrix elements a_{ij}.

14.6. Do the same as in the previous exercise with c_0 in place of ℓ^1.

14.7. Let Y be a Banach space and $Z \subset Y$ a closed vector subspace. Suppose that there exists a Banach space X such that the map $Z \mathbin{\widehat{\otimes}_\pi} X' \to Y \mathbin{\widehat{\otimes}_\pi} X'$ generated by the inclusion $Z \hookrightarrow Y$ is not topologically injective (cf. Exercises 13.6–13.8). Assume also that the canonical map $Y \mathbin{\widehat{\otimes}_\pi} X' \to \mathscr{L}(X, Y)$ is injective. Prove that there exists a nuclear operator from X to Y whose range is contained in Z but which is not a nuclear operator from X to Z.

14.8. Construct Fréchet spaces X, Y and a non-nuclear $\varphi\colon X \to Y$ which belongs to the range of the canonical map $Y \mathbin{\widehat{\otimes}_\pi} X'_\beta \to \mathscr{L}(X, Y)$. (Thus the tensor product definition of a nuclear operator between Banach spaces is not appropriate for Fréchet spaces.)

14.9. Let M be a complex manifold (for simplicity, you may assume that M is an open subset of $\mathbb{C}$), and let U be an open, relatively compact subset of M. Show that the restriction map $\mathscr{O}(M) \to \mathscr{O}(U)$ is nuclear.

Definition 14.1. A linear operator $\varphi\colon X \to Y$ between locally convex spaces is *compact* if there exists a 0-neighborhood $U \subset X$ such that $\varphi(U)$ is relatively compact in Y.

14.10. Show that **(a)** a compact operator between locally convex spaces is continuous;
(b) a nuclear operator between locally convex spaces is compact.

14.11. Let X and Y be vector spaces, $\varphi\colon X \to Y$ a linear map, and $B \subset X$ a Banach disk. Show that $\varphi(B)$ is a Banach disk in Y.

14.12. Let X be a locally convex space, and let $B \subset X$ be an absolutely convex bounded set. Show that
(a) the inclusion $j_B \colon X_B \to X$ is continuous;
(b) if X is Hausdorff, then X_B is a normed space;
(c) if X is Hausdorff and B is complete, then B is a Banach disk.

14.13. Let X be a locally convex space, and let p be a continuous seminorm on X. Consider the set $B_p = \{f \in X' : |f(x)| \leqslant 1 \ \forall x \in U_p\}$. Show that B_p is a Banach disk, and construct an isometric isomorphism $(X')_{B_p} \cong (\widetilde{X}_p)'$.

14.14. Let X and Y be locally convex spaces. Show that each nuclear operator $\varphi \colon X \to Y$ uniquely extends to a nuclear operator $\widetilde{\varphi} \colon \widetilde{X} \to Y$.

14.15. Let $\varphi_1 \colon X_1 \to Y_1$ and $\varphi_2 \colon X_2 \to Y_2$ be nuclear operators between locally convex spaces.
(a) Show that $\varphi_1 \mathbin{\widehat{\otimes}_\pi} \varphi_2 \colon X_1 \mathbin{\widehat{\otimes}_\pi} X_2 \to Y_1 \mathbin{\widehat{\otimes}_\pi} Y_2$ is nuclear.
(b) Is $\varphi_1 \otimes_\pi \varphi_2 \colon X_1 \otimes_\pi X_2 \to Y_1 \otimes_\pi Y_2$ necessarily nuclear?

14.16. Let $\varphi \colon X \to Y$ be a nuclear operator between locally convex spaces. Show that the dual operator $\varphi' \colon Y'_\beta \to X'_\beta$ is nuclear.