

Köthe sequence spaces

(EXERCISES FOR LECTURE 14)

12.1. Prove that every Köthe space $\lambda^\nu(I, P)$ is complete.

12.2. Let P be a Köthe set on a set I , and let $\nu \in [1, +\infty]$.

(a) Assume that $p_i > 0$ for all $p \in P$ and all $i \in I$. Construct a topological isomorphism between $\lambda^\nu(I, P)$ and the projective limit of a family of the Banach spaces $\ell^\nu(I)$ where the connecting maps of the projective system are diagonal operators.

(b) Extend (a) to an arbitrary Köthe set P .

12.3. Consider the Köthe set $P = \{p^{(1)}, p^{(2)}, \dots\}$ on \mathbb{N} , where $p^{(m)} = (1, \dots, 1, 0, 0, \dots)$ (the first m entries are equal to 1, the other entries are 0).

(a) Show that for every $\nu \in [1, +\infty] \cup \{0\}$ we have $\lambda^\nu(\mathbb{N}, P) = \mathbb{K}^{\mathbb{N}}$ topologically.

(b) Extend (a) to the space \mathbb{K}^S , where S is an arbitrary set.

12.4. Consider the Köthe set $P = \{p^{(1)}, p^{(2)}, \dots\}$ on \mathbb{N} , where $p_k^{(m)} = k^m$. Show that for every $\nu \in [1, +\infty] \cup \{0\}$ we have $\lambda^\nu(\mathbb{N}, P) = s$ topologically. (Recall that $\lambda^\infty(\mathbb{N}, P) = s$ by definition, see Exercise sheet 2.)

12.5. Consider the Köthe set $P = \{p^{(1)}, p^{(2)}, \dots\}$ on $\mathbb{Z}_{\geq 0}$, where $p_k^{(m)} = m^k$. Show that for every $\nu \in [1, +\infty] \cup \{0\}$ there is a topological isomorphism $\lambda^\nu(\mathbb{Z}_{\geq 0}, P) \cong \mathcal{O}(\mathbb{C})$.

12.6. Consider the Köthe set $P = \{p^{(1)}, p^{(2)}, \dots\}$ on \mathbb{Z}^n , where $p_k^{(m)} = (1 + |k|)^m$. Show that for every $\nu \in [1, +\infty] \cup \{0\}$ we have $\lambda^\nu(\mathbb{Z}^n, P) = s(\mathbb{Z}^n)$ topologically. (We have $\lambda^\infty(\mathbb{Z}^n, P) = s(\mathbb{Z}^n)$ by definition, cf. Exercise sheet 2.)

12.7. Given $R = (R_1, \dots, R_n) \in (0, +\infty]^n$, define a Köthe set P on $\mathbb{Z}_{\geq 0}^n$ by letting

$$P = \{p^{(r)} : r = (r_1, \dots, r_n) \in (0, R_1) \times \dots \times (0, R_n)\},$$

where $p_k^{(r)} = r_1^{k_1} \dots r_n^{k_n}$ for $k = (k_1, \dots, k_n) \in \mathbb{Z}_{\geq 0}^n$. Show that for every $\nu \in [1, +\infty] \cup \{0\}$ there is a topological isomorphism $\lambda^\nu(\mathbb{Z}_{\geq 0}^n, P) \cong \mathcal{O}(\mathbb{D}_R^n)$, where $\mathbb{D}_R^n = \{z \in \mathbb{C}^n : |z_i| < R_i \ \forall i = 1, \dots, n\}$ is the open polydisk in \mathbb{C}^n of polyradius R .

12.8 (*the Aizenberg-Mityagin theorem*). An open set D in \mathbb{C}^n is a *complete Reinhardt domain* if for each $z = (z_1, \dots, z_n) \in D$ and each $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that $|\lambda_i| \leq 1$ ($i = 1, \dots, n$) we have $(\lambda_1 z_1, \dots, \lambda_n z_n) \in D$. Given a complete bounded Reinhardt domain D and $k = (k_1, \dots, k_n) \in \mathbb{Z}_{\geq 0}^n$, let $b_k(D) = \sup_{z \in D} |z_1^{k_1} \dots z_n^{k_n}|$. Define a Köthe set P on $\mathbb{Z}_{\geq 0}^n$ by letting

$$P = \{(b_k(D) s^{|k|})_{k \in \mathbb{Z}_{\geq 0}^n} : 0 < s < 1\}$$

where $|k| = k_1 + \dots + k_n$ for $k = (k_1, \dots, k_n) \in \mathbb{Z}_{\geq 0}^n$.

(a) Show that for every $\nu \in [1, +\infty] \cup \{0\}$ there is a topological isomorphism $\lambda^\nu(\mathbb{Z}_{\geq 0}^n, P) \cong \mathcal{O}(D)$.

(b) Show that, if $D = \mathbb{D}_R^n$ is the open polydisk of polyradius R , then (a) yields the result of Exercise 12.7.

(c) Give an explicit form of (a) in the case where $D = \mathbb{B}_R^n = \{z \in \mathbb{C}^n : \sum |z_i|^2 < R^2\}$ is the open ball of radius R .

12.9. Given $z \in \mathbb{C}$, construct a topological isomorphism $\mathcal{O}_z \cong \lambda^\nu(\mathbb{Z}_{\geq 0}, P)$, where P is a suitable Köthe set on $\mathbb{Z}_{\geq 0}$, and where $\nu \in [1, +\infty] \cup \{0\}$ is arbitrary. As a corollary, \mathcal{O}_z is complete. (*Hint:* see Exercise 8.7 (a).)

12.10. Consider the vector space of *tempered sequences*

$$X = \left\{ x = (x_n) \in \mathbb{K}^{\mathbb{N}} : \sup_{n \in \mathbb{N}} |x_n| n^{-k} < \infty \text{ for some } k \in \mathbb{N} \right\}.$$

We equip X with the locally convex inductive limit topology of the sequence $(X_k)_{k \in \mathbb{N}}$ of Banach spaces, where

$$X_k = \left\{ x = (x_n)_{n \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \|x\|_k = \sup_{n \in \mathbb{N}} |x_n| n^{-k} < \infty \right\}$$

and the connecting maps $X_k \rightarrow X_\ell$ ($k \leq \ell$) are the tautological inclusions.

(a) Let s' denote the strong dual of the space s of rapidly decreasing sequences. Show that the map

$$s' \rightarrow X, \quad F \mapsto (x_n(F) = F(e_n))_{n \in \mathbb{N}},$$

is a topological isomorphism (here e_n is the sequence with 1 in the n th slot, 0 elsewhere).

(b) Let P denote the family of all nonnegative sequences from s . Show that, for each $\nu \in [1, +\infty] \cup \{0\}$, we have $X = \lambda^\nu(\mathbb{N}, P)$ topologically.

12.11. Consider the vector space

$$X = \left\{ x = (x_n) \in \mathbb{C}^{\mathbb{Z}_{\geq 0}} : \sup_{n \in \mathbb{N}} |x_n| r^{-n} < \infty \text{ for some } r > 0 \right\}.$$

We equip X with the locally convex inductive limit topology of the family $(X_r)_{r > 0}$ of Banach spaces, where

$$X_r = \left\{ x = (x_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{Z}_{\geq 0}} : \|x\|_r = \sup_{n \in \mathbb{N}} |x_n| r^{-n} < \infty \right\}$$

and the connecting maps $X_r \rightarrow X_s$ ($r \geq s$) are the tautological inclusions.

(a) Let $\mathcal{O}(\mathbb{D})'$ denote the strong dual of the space $\mathcal{O}(\mathbb{D})$ of holomorphic functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Show that the map

$$\mathcal{O}(\mathbb{D})' \rightarrow X, \quad F \mapsto (x_n(F) = F(z^n/n!))_{n \in \mathbb{Z}_{\geq 0}},$$

is a topological isomorphism.

(b) Let P denote the family of all nonnegative sequences $p = (p_n)_{n \geq 0}$ such that the sequence $(p_n r^n)$ is bounded for all $r \in (0, 1)$ (equivalently, such that the power series $\sum_n p_n z^n$ converges in \mathbb{D}). Show that, for each $\nu \in [1, +\infty] \cup \{0\}$, we have $X = \lambda^\nu(\mathbb{Z}_{\geq 0}, P)$ topologically.