Topological tensor products I

(exercises for Lectures 12–13)

11.1. Let X, Y, Z be seminormed spaces. Show that
(a) a bilinear map $\Phi\colon X \times Y \to Z$ is continuous iff $\|\Phi\| < \infty$, where

$$\|\Phi\| = \sup\{\|\Phi(x,y)\| : \|x\| \leqslant 1,\ \|y\| \leqslant 1\}; \tag{1}$$

(b) we have $\|\Phi\| \leqslant C$ (where $0 \leqslant C < +\infty$) iff $\Phi(U_X \times U_Y) \subset CU_Z$, where U_X, U_Y, U_Z are the open (or closed) unit balls of X, Y, Z;
(c) the operator seminorm (1) is indeed a seminorm on the space $\mathscr{L}^2(X \times Y, Z)$ of continuous bilinear maps;
(d) $\mathscr{L}^2(X \times Y, Z)$ is a normed space iff Z is a normed space.

11.2. State and prove (or recall from your functional analysis course) an analog of Exercise 11.1 for linear maps.

11.3. Let X, Y, Z be locally convex spaces, and let P, Q, R be defining families of seminorms on X, Y, Z, resp. Suppose that P and Q are directed. Let $\Phi\colon X \times Y \to Z$ be a bilinear map. Show that the following conditions are equivalent:

- (i) Φ is continuous;
- (ii) for every $r \in R$ there exist $p \in P$, $q \in Q$, and $C > 0$ such that $r(\Phi(x,y)) \leqslant Cp(x)q(y)$ for all $x \in X$, $y \in Y$;
- (iii) for every $r \in R$ there exist $p \in P$ and $q \in Q$ such that $\Phi\colon X_p \times Y_q \to Z_r$ is continuous (where $X_p = (X, p)$ etc.).

11.4. Let X and Y be seminormed spaces. Show that the open unit ball of $X \otimes_\pi Y$ is the convex hull of the set $U_X \odot U_Y = \{x \otimes y : x \in U_X,\ y \in U_Y\}$, where U_X and U_Y are the open unit balls of X and Y, respectively. As a corollary, the projective tensor seminorm on $X \otimes Y$ is the Minkowski functional of $\operatorname{conv}(U_X \odot U_Y)$.

11.5. Let X and Y be locally convex spaces. Show that
(a) the topology on $X \otimes_\pi Y$ is the finest locally convex topology on $X \otimes Y$ making the canonical map $X \times Y \to X \otimes Y$, $(x, y) \mapsto x \otimes y$, continuous;
(b) if β_X and β_Y are bases of neighborhoods of 0 in X and Y, respectively, then $\{\operatorname{conv}(U \odot V) : U \in \beta_X,\ V \in \beta_Y\}$ is a base of neighborhoods of 0 in $X \otimes_\pi Y$.

11.6. Let X and Y be infinite-dimensional normed spaces. Prove that the normed spaces $X \otimes_\pi Y$ and $X \otimes_\varepsilon Y$ are incomplete.

11.7. State and prove **(a)** the commutativity and **(b)** the associativity of the tensor products $\otimes_\pi$, $\otimes_\varepsilon$, $\widehat{\otimes}_\pi$, $\widehat{\otimes}_\varepsilon$, and **(c)** their additivity in each variable.

11.8. Given seminormed spaces X, Y, Z, construct natural isometric isomorphisms $\mathscr{L}(X \otimes_\pi Y, Z) \cong \mathscr{L}(X, \mathscr{L}(Y, Z))$ and (assuming that Z is a Banach space) $\mathscr{L}(X \mathbin{\widehat{\otimes}_\pi} Y, Z) \cong \mathscr{L}(X, \mathscr{L}(Y, Z))$.

11.9. Given locally convex spaces X, Y, Z, construct a natural linear injection $\mathscr{L}(X \otimes_\pi Y, Z) \hookrightarrow \mathscr{L}(X, \mathscr{L}_b(Y, Z))$. Give an example showing that this map is not necessarily surjective. (*Hint:* take an infinite-dimensional Banach space X, let $Z = \mathbb{K}$, and try to guess what Y is.)

11.10. Let X be a locally convex space, and $\{Y_i : i \in I\}$ be a family of locally convex spaces.
(a) Is the natural vector space isomorphism $X \otimes_\pi (\bigoplus_{i\in I} Y_i) \cong \bigoplus_{i\in I}(X \otimes_\pi Y_i)$ always a topological isomorphism? **(b)** The same question for $\otimes_\varepsilon$.

11.11. Given a normed space X, consider the normed spaces $X_1^n = (X^n, \|\cdot\|_1)$ and $X_\infty^n = (X^n, \|\cdot\|_\infty)$, where $\|x\|_1 = \sum \|x_i\|$ and $\|x\|_\infty = \max \|x_i\|$ for $x = (x_1, \ldots, x_n) \in X^n$.
(a) Construct isometric isomorphisms $\mathbb{K}_1^n \otimes_\pi X \cong X_1^n$ and $\mathbb{K}_\infty^n \otimes_\varepsilon X \cong X_\infty^n$.
(b) Identify $\mathbb{K}_1^n \otimes \mathbb{K}_\infty^n$ with the space $M_n(\mathbb{K})$ of $n \times n$-matrices via the isomorphism $x \otimes y \mapsto (x_i y_j)$. Given $a = (a_{ij}) \in M_n(\mathbb{K})$, calculate $\|a\|_\pi$ and $\|a\|_\varepsilon$ explicitly in terms of the matrix elements a_{ij}, and deduce that $\|\cdot\|_\pi \neq \|\cdot\|_\varepsilon$ unless $n = 1$.

11.12. Given a set I and a Banach space X, construct an isometric isomorphism $\ell^1(I) \widehat{\otimes}_\pi X \cong \ell^1(I, X)$, where

$$\ell^1(I, X) = \Big\{x = (x_i) \in X^I : \|x\| = \sum_i \|x_i\| < \infty\Big\}.$$

11.13. Fill in the details of the construction of the isometric isomorphism

$$L^1(X, \mu) \widehat{\otimes}_\pi L^1(Y, \nu) \cong L^1(X \times Y, \mu \times \nu),$$

where (X, μ) and (Y, ν) are measure spaces (see the lecture).