Equicontinuity. Barrelled and bornological spaces

(EXERCISES FOR LECTURES 10–11)

10.1. Let X and Y be l.c.s.'s, and let $S \subset \mathscr{L}(X, Y)$ be an equicontinuous set. Show that
(a) the topology of pointwise convergence on elements of any dense subset $D \subset X$ and the topology of uniform convergence on compact subsets of X induce the same topology on S;
(b) if X is separable and Y is metrizable, then S is metrizable for the topology of pointwise convergence.

10.2. Let X and Y be l.c.s.'s. Suppose that X is barrelled.
(a) Show that, if (φ_n) is a sequence in $\mathscr{L}(X, Y)$ which converges pointwise to a map $\varphi \colon X \to Y$, then $\varphi \in \mathscr{L}(X, Y)$. Deduce that, if Y is sequentially complete, then so is $\mathscr{L}_s(X, Y)$.
(b) Does a similar statement hold for nets? Does the completeness of Y imply the completeness of $\mathscr{L}_s(X, Y)$? (*Hint*: see Exercise 9.18.)

10.3. Let X and Y be l.c.s.'s. Show that, if X is quasibarrelled, then each bounded subset of $\mathscr{L}_b(X, Y)$ is equicontinuous.

10.4 (*a nonquasibarrelled space*)**.** Let X be an infinite-dimensional normed space, and let X_σ denote X equipped with the weak topology. Show that the unit ball in $X' = (X_\sigma)'$ is bounded for the topology of uniform convergence on bounded subsets of X_σ, but is not equicontinuous as a subset of $(X_\sigma)'$. Deduce that X_σ is not quasibarrelled.

10.5 (*a nonbarrelled normed space*)**.** Let c_{00} denote the space of all finite sequences in $\mathbb{K}$ (i.e., of those $x = (x_n) \in \mathbb{K}^{\mathbb{N}}$ for which $x_n = 0$ for all but finitely many n). Let $X = (c_{00}, \|\cdot\|_p)$, where $\|\cdot\|_p$ is the norm induced from ℓ^p, $1 \leqslant p \leqslant \infty$.
(a) Construct a sequence (f_n) in X' which converges to 0 pointwise, but which is unbounded for the standard norm on X'. Deduce that X is not barrelled.
(b) Construct explicitly a barrel in X which is not a neighborhood of 0.

10.6. (a) Show that a hyperplane (i.e., a codimension 1 vector subspace) of a barrelled l.c.s. is barrelled.
(b) Deduce that there exists an incomplete barrelled normed space.

10.7* (*a nonbornological barrelled space*)**.** Given an uncountable set S, let $X = \mathbb{K}^S$, and let X_0 be the subspace of X consisting of all countably supported functions. Let also $X_1 = X_0 \oplus \mathbb{K}1 \subset X$. Show that
(a) X and X_0 are barrelled;
(b) if an l.c.s. Y has a dense barrelled vector subspace, then Y is barrelled;
(c) X_1 is barrelled;
(d) a sequentially closed hyperplane of a bornological l.c.s. is closed;
(e) X_1 is not bornological.

10.8. Construct normed spaces X, Y and a separately continuous bilinear map $f \colon X \times Y \to \mathbb{K}$ which is not continuous.

10.9. Let X (resp., Y) be the product (resp., the locally convex direct sum) of countably many copies of $\mathbb{K}$. Show that the pairing $X \times Y \to \mathbb{K}$, $(x, y) \mapsto \sum_i x_i y_i$, is separately continuous, but is not continuous.

Definition 10.1. Let X be an l.c.s. A *bornological space associated to* X is a bornological l.c.s. X_{born} together with $i \in \mathscr{L}(X_{\mathrm{born}}, X)$ such that, for each bornological l.c.s. Y and each $\varphi \in \mathscr{L}(Y, X)$, there exists a unique $\psi \in \mathscr{L}(Y, X_{\mathrm{born}})$ satisfying $i \circ \psi = \varphi$.

10.10. Give a functorial interpretation of X_{born}. State and prove a uniqueness theorem for X_{born}.

10.11. Given an l.c.s. X, let $\mathcal{T}_{\mathrm{born}}$ denote the family of all bornological locally convex topologies on X which are finer than the original topology on X. Let X_{born} denote the space X equipped with the inductive locally convex topology generated by the identity maps $\{(X, \tau) \to X : \tau \in \mathcal{T}_{\mathrm{born}}\}$. Show that X_{born} together with the identity map $X_{\mathrm{born}} \to X$ is a bornological space associated to X.

10.12. Given an l.c.s. X, define barrelled and the quasibarrelled spaces associated to X, give functorial interpretations for them, and prove their existence and uniqueness.

10.13 (*another construction of* X_{born})**.** Given an l.c.s. X, let P_{born} denote the family of all seminorms on X which take bounded subsets of X to bounded subsets of $\mathbb{R}$. Show that $(X, \tau(P_{\mathrm{born}}))$ together with the identity map $(X, \tau(P_{\mathrm{born}})) \to X$ is a bornological space associated to X.

10.14 (*yet another construction of* X_{born})**.** Given an l.c.s. X, let $\mathrm{Bdd}(X)$ denote the family of all bounded, absolutely convex subsets of X. Given $B_1, B_2 \in \mathrm{Bdd}(X)$, we let $B_1 \leqslant B_2$ if B_1 is absorbed by B_2. For each $B \in \mathrm{Bdd}(X)$ we let $X_B = \mathrm{span}(B)$, and we make X_B into a seminormed space w.r.t. the Minkowski functional of B. Show that

(a) $\mathrm{Bdd}(X)$ is a directed set, and $\{X_B : B \in \mathrm{Bdd}(X)\}$ is an inductive system of seminormed spaces w.r.t. the inclusion maps $X_{B_1} \to X_{B_2}$ $(B_1 \leqslant B_2)$;

(b) $\varinjlim\{X_B : B \in \mathrm{Bdd}(X)\} \cong X_{\mathrm{born}}$.

10.15. Show that

(a) an l.c.s. is bornological iff it is isomorphic to an inductive limit of seminormed spaces;

(b) a complete bornological l.c.s. is isomorphic to an inductive limit of Banach spaces.

10.16. Let X and Y be metrizable l.c.s.'s. Suppose that X is complete. Given $\varphi \in \mathscr{L}(X, Y)$, show that either $\varphi(X) = Y$, or $\varphi(X)$ is meagre in Y (i.e., is a union of countably many nowhere dense sets).

10.17 (*Grothendieck's factorization theorem*)**.** Let X be a Fréchet space, (X_n) be a sequence of Fréchet spaces, and Y be an l.c.s. Suppose we have a continuous linear map $\varphi\colon X \to Y$ and a sequence $(\varphi_n\colon X_n \to Y)_{n\in\mathbb{N}}$ of continuous linear maps such that $\varphi(X) \subset \bigcup_n \varphi_n(X_n)$. Show that

(a) $\varphi(X) \subset \varphi_n(X_n)$ for some n;

(b) if all the maps φ_i are injective, then φ can be decomposed as $\varphi = \varphi_n \psi$ for some n and for some continuous linear map $\psi\colon X \to X_n$.

Hint. Let $Z_n = X \times_Y X_n$, and let $\pi_n\colon Z_n \to X$ denote the standard projection. By using Baire's theorem and Exercise 10.16, show that $X = \pi_n(Z_n)$ for some n.

10.18 (*Grothendieck's open mapping and closed graph theorems*)**.** Let X and Y be l.c.s.'s. Suppose that X is equipped with the inductive locally convex topology generated by a countable family $(\varphi_n\colon X_n \to X)_{n\in\mathbb{N}}$ of linear maps, where X_n are Fréchet spaces and $\bigcup_n \varphi_n(X_n) = X$. Suppose also that Y is equipped with the inductive locally convex topology generated by a (not necessarily countable) family $(\psi_i\colon Y_i \to Y)_{i\in I}$ of linear maps, where Y_i are Fréchet spaces (for example, this is true provided that Y is a complete bornological space; see Exercise 10.15). Show that

(a) each surjective continuous linear map of X onto Y is open;

(b) each linear map from Y to X whose graph is closed in $Y \times X$ is continuous.