Completeness

(EXERCISES FOR LECTURES 8–10)

9.1. Let X be a topological space, $Y \subset X$, and $x \in X$. Show that $x \in \overline{Y}$ iff there is a net in Y which converges to x.

9.2. Let X and Y be topological spaces. Show that a map $f\colon X \to Y$ is continuous at $x \in X$ iff for every net (x_λ) in X such that $x_\lambda \to x$ we have $f(x_\lambda) \to f(x)$ in Y.

9.3. Show that a topological space X is Hausdorff iff every net in X has at most one limit.

9.4. Show that a topological space X is compact iff every net in X has an accumulation point.

9.5. Let X be a topological vector space. Prove that
(a) each convergent net in X is a Cauchy net;
(b) each Cauchy net in X which has an accumulation point $x \in X$ converges to x.

9.6. Show that a continuous linear map between topological vector spaces takes Cauchy nets to Cauchy nets.

9.7. Let $(X, \tau(P))$ be a locally convex space. Show that a net (x_λ) in X
(a) converges to $x \in X$ iff $p(x_\lambda - x) \to 0$ for every $p \in P$;
(b) is a Cauchy net iff for each $p \in P$ and each $\varepsilon > 0$ there exists $\lambda_0 \in \Lambda$ such that we have $p(x_\lambda - x_\mu) < \varepsilon$ whenever $\lambda, \mu \geqslant \lambda_0$.

9.8. Let X be a vector space equipped with the projective locally convex topology generated by a family $\{\varphi_i\colon X \to X_i : i \in I\}$ of linear maps, where $\{X_i : i \in I\}$ is a family of locally convex spaces. Show that a net (x_λ) in X converges to $x \in X$ iff we have $\varphi_i(x_\lambda) \to \varphi_i(x)$ for all $i \in I$.

9.9. Show that a compact subset of a Hausdorff topological vector space is complete.

9.10. Let X be a metrizable topological vector space, and let ρ be a translation invariant metric on X that generates the topology of X. Prove that the following conditions are equivalent:

(i) X is complete;
(ii) X is sequentially complete;
(iii) (X, ρ) is a complete metric space.

9.11. Let S be an uncountable set, and let X be the subspace of $\mathbb{K}^S$ consisting of all countably supported functions. Prove that X is sequentially complete, but is not complete.

9.12. Let X be a topological vector space, and let $X_0 \subset X$ be a dense vector subspace. Show that
(a) every continuous seminorm p on X_0 uniquely extends to a continuous seminorm $\tilde{p}$ on X;
(b) if P is a defining family of seminorms on X_0, then $\{\tilde{p} : p \in P\}$ is a defining family of seminorms on X;
(c) if $\mathscr{U}$ is a base of neighborhoods of 0 in X_0, then $\{\overline{U} : U \in \mathscr{U}\}$ is a base of neighborhoods of 0 in X.

9.13. Let $(X_i)_{i \in I}$ be a family of locally convex spaces. Show that $\prod_{i \in I} X_i$ is complete iff all the spaces X_i are complete.

9.14. Let $(X_i)_{i\in I}$ be a family of locally convex spaces. Show that $\bigoplus_{i\in I} X_i$ is complete iff all the spaces X_i are complete. As a corollary, the strongest locally convex space is complete.

9.15. Let X be a locally convex space, and let Y be a vector subspace of X. Suppose that Y is equipped with a locally convex topology that is stronger (=finer) than the topology induced from X. We say that Y is *locally closed* in X if there is a base of neighborhoods of 0 in Y consisting of sets closed in X. Show that, if X is complete and Y is locally closed in X, then Y is complete.

9.16. Prove that the following locally convex spaces are complete:
(a) $C(T)$, where T is a locally compact topological space;
(b) $C^\infty(U)$, where $U \subset \mathbb{R}^n$ is an open set;
(c) the space s of rapidly decreasing sequences;
(d) the Schwartz space $\mathscr{S}(\mathbb{R}^n)$;
(e) $\mathscr{O}(U)$, where $U \subset \mathbb{C}$ is an open set;
(f) $C_c(T)$, where T is a second countable locally compact topological space;
(g) $C_c^\infty(U)$, where $U \subset \mathbb{R}^n$ is an open set;
(h) the space $\mathscr{O}_z$ of holomorphic germs at $z \in \mathbb{C}$.
Hint to **(h)**: $\mathscr{O}_z \cong \varprojlim_{p\in P} \ell^1(\mathbb{N}, p)$, where P is as in Exercise 8.7 and $\ell^1(\mathbb{N}, p)$ is the respective weighted ℓ^1-space (i.e., $\ell^1(\mathbb{N}, p)$ consists of sequences $x = (x_n) \in \mathbb{K}^{\mathbb{N}}$ such that $\|x\|_p = \sum_n |x_n| p_n < \infty$).

9.17. Let X be a complete locally convex space, and let P be a directed defining family of seminorms on X. Recall (see the lecture) that there exists a topological isomorphism $X \cong \varprojlim \tilde{X}_p$, where $\tilde{X}_p$ (for every $p \in P$) is the completion of the normed space $X_p^0 = (X/p^{-1}(0), \hat{p})$. Describe $\tilde{X}_p$ explicitly for **(0)** $X = \mathbb{K}^S$, where S is a set, and for the spaces **(a)**–**(f)** of Exercise 9.16.

9.18. Let X be a Hausdorff locally convex space. Describe explicitly the completion of the dual space X' equipped with the weak* topology.

9.19. Given a locally convex space X, let X^∞ (resp. X_∞) denote the product (resp. the locally convex direct sum) of countably many copies of X. Let now X and Y be Banach spaces such that Y is continuously embedded into X and such that Y is dense in X (for example, $X = \ell^2$ and $Y = \ell^1$). Define

$$\varphi\colon X_\infty \oplus Y^\infty \to X^\infty, \quad (x, y) \mapsto x + y.$$

Prove that φ is an open map onto a proper dense subspace of X^∞. Deduce that the quotient $(X_\infty \oplus Y^\infty)/\operatorname{Ker}\varphi$ is incomplete (while $X_\infty \oplus Y^\infty$ itself is complete).

9.20. Let T be a locally compact Hausdorff topological space, and let $S \subset T$ be a closed subset.
(a) Prove that

(i) the restriction map $r_S\colon C(T) \to C(S)$ is an open map onto a dense subspace of $C(S)$;
(ii) if T is not normal, then there exists a closed set $S \subset T$ such that r_S is not onto;
(iii) if S is as in (ii), then the quotient $C(T)/\operatorname{Ker} r_S$ is incomplete (while $C(T)$ itself is complete).

(b) Let $T = [0, \omega_1]^2 \setminus \{(\omega_1, \omega_1)\}$, where ω_1 is the first uncountable ordinal. Show that T is locally compact, but is not normal. Find a concrete $S \subset T$ satisfying (ii).