

## Inductive limits

(EXERCISES FOR LECTURES 7–8)

**8.1.** Let  $(X_i, \varphi_{ij})$  be an inductive system of locally convex spaces indexed by a directed set  $I$ . Let

$$X = \left( \bigoplus_{i \in I} X_i \right) / \text{span}\{x - \varphi_{ij}(x) : i \leq j, x \in X_i\}.$$

For each  $j \in I$ , let  $\varphi_j: X_j \rightarrow X$  denote the composite of the standard embedding  $X_j \rightarrow \bigoplus_{i \in I} X_i$  and the quotient map  $\bigoplus_{i \in I} X_i \rightarrow X$ . Show that  $(X, \varphi_i)$  is a (category-theoretic) inductive limit of  $(X_i, \varphi_{ij})$  in LCS.

**8.2** (*directed families of subspaces*). Let  $X$  be a vector space, and let  $(X_i)_{i \in I}$  be a family of vector subspaces of  $X$  indexed by a directed set  $I$  in such a way that  $X_i \subset X_j$  whenever  $i \leq j$ . Let  $\varphi_{ij}: X_i \hookrightarrow X_j$  denote the respective inclusion maps.

(a) Show that  $Y = \bigcup_i X_i$  together with the inclusion maps  $(\varphi_i: X_i \hookrightarrow Y)_{i \in I}$  is an inductive limit of  $(X_i, \varphi_{ij})$  in the category of vector spaces.

(b) Suppose also that each  $X_i$  is equipped with a locally convex topology in such a way that the inclusions  $\varphi_{ij}$  are continuous for all  $i \leq j$ . Equip  $Y$  with the inductive locally convex topology generated by the inclusion maps  $\varphi_i: X_i \hookrightarrow Y$ . Show that  $Y$  together with  $(\varphi_i)_{i \in I}$  is an inductive limit of  $(X_i, \varphi_{ij})$  in LCS.

**8.3.** Let  $F = (X_i, \varphi_{ij})$  be an inductive system of vector spaces indexed by a directed set  $I$ , let  $X = \varinjlim X_i$ , and let  $\varphi_i: X_i \rightarrow X$  denote the canonical maps. Show that, for each  $i \in I$ , we have  $\text{Ker } \varphi_i = \bigcup_{j \geq i} \text{Ker } \varphi_{ij}$ . Deduce that, if all  $\varphi_{ij}$  are injective, then  $F$  is isomorphic (as an inductive system) to a suitable directed family of subspaces of  $X$  (see Exercise 8.2 (a)).

**8.4.** Let  $K$  be a compact subset of  $\mathbb{C}^n$ , and let  $\mathcal{U}$  be a base of relatively compact open neighborhoods of  $K$ . Recall (see the lecture) that the space  $\mathcal{O}(K)$  of germs of holomorphic functions on  $K$  is the locally convex inductive limit  $\varinjlim \{\mathcal{O}(U) : U \in \mathcal{U}\}$ . Construct a topological isomorphism  $\mathcal{O}(K) \cong \varinjlim \{\mathcal{A}(\overline{U}) : U \in \mathcal{U}\}$ , where  $\mathcal{A}(\overline{U})$  is defined in Exercise 6.9.

**8.5.** Let  $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ . Show that the restriction map  $\mathcal{O}(\mathbb{D}_R) \rightarrow \mathcal{O}(\mathbb{D}_r)$  (where  $R > r$ ) is not topologically injective. As a corollary, the inductive sequence  $\{\mathcal{O}(\mathbb{D}_{1/n}) : n \in \mathbb{N}\}$  is not strict.

**8.6.** Prove that the following spaces are Hausdorff: (a)  $C_c(X)$  (where  $X$  is a topological space); (b)  $C_c^\infty(U)$  (where  $U$  is an open subset of  $\mathbb{R}^n$ ); (c)  $\mathcal{O}(K)$  (where  $K$  is a compact subset of  $\mathbb{C}^n$ ). (In (a) and (b), try to avoid using general results on strict inductive limits. In (c), the respective inductive limit is not strict by Exercise 8.5.)

**8.7.** Given  $z \in \mathbb{C}$ , let  $\mathcal{O}_z$  denote the space of germs of holomorphic functions at  $z$  (i.e.,  $\mathcal{O}_z = \mathcal{O}(\{z\})$ ), see Exercise 8.4).

(a) Let  $P$  denote the set of all sequences  $p = (p_n)$ ,  $p_n > 0$ , such that  $p_n = o(\varepsilon^n)$  for every  $\varepsilon > 0$ . For each  $p \in P$ , define a norm  $\|\cdot\|_p$  on  $\mathcal{O}_z$  by  $\|f\|_p = \sum_n |c_n(f)| p_n$ , where  $c_n(f) = f^{(n)}(z)/n!$ . Show that the family  $\{\|\cdot\|_p : p \in P\}$  of norms is defining for  $\mathcal{O}_z$ .

(b) Show that a sequence  $(f_n)$  converges in  $\mathcal{O}_z$  iff there is a neighborhood  $U \ni z$  such that  $(f_n)$  is contained in  $\mathcal{O}(U)$  and converges uniformly on  $U$ .

(c) Is  $\mathcal{O}_z$  metrizable?

**8.8.** Let  $\mathcal{C}_0 = \varinjlim C[-1/n, 1/n]$  be the space of germs of continuous functions at  $0 \in \mathbb{R}$  equipped with the respective inductive locally convex topology, and let  $E = \{f \in \mathcal{C}_0 : f(0) = 0\}$ . Prove that the topology on  $E$  induced from  $\mathcal{C}_0$  is anti-discrete. As a corollary,  $\mathcal{C}_0$  is not Hausdorff.

**8.9.** Let  $\mathcal{O}_0$  denote the space of germs of holomorphic functions at  $0 \in \mathbb{C}$ . Write  $\mathcal{O}_0$  in the form  $\mathcal{O}_z = \varinjlim \mathcal{O}(\mathbb{D}_{1/n})$ , where  $\mathbb{D}_{1/n} = \{z \in \mathbb{C} : |z| < 1/n\}$ . Let us now change the canonical topology on  $\mathcal{O}_0$  as follows: equip each  $\mathcal{O}(\mathbb{D}_{1/n})$  with the topology of pointwise convergence, and equip  $\mathcal{O}_0$  with the respective inductive locally convex topology. Let  $E = \{f \in \mathcal{O}_0 : f(0) = 0\}$ . Prove that the topology on  $E$  induced from  $\mathcal{O}_0$  is anti-discrete. As a corollary,  $\mathcal{O}_0$  is not Hausdorff for the above (nonstandard) topology.

**8.10.** Let  $X$  be the inductive limit of a strict inductive sequence  $(X_n)$  of locally convex spaces. Suppose that each  $X_n$  is closed in  $X_{n+1}$ . Prove that  $B \subset X$  is bounded iff there exists  $n$  such that  $B \subset X_n$  and such that  $B$  is bounded in  $X_n$ .

**8.11.** Let  $Y$  be a normed space, and let  $(Y_n)$  be a strictly increasing sequence of vector subspaces of  $Y$  such that  $Y_n$  is dense in  $Y_{n+1}$  for each  $n$ . Let  $Z = \bigcup_n Y_n$ .

(a) Give a concrete example of such a sequence  $(Y_n)$ .

(b) Describe (as explicitly as you can) the inductive locally convex topology on  $Z$  generated by the inclusion maps  $Y_n \hookrightarrow Z$ .

(c) Does the result of Exercise 8.10 hold without the assumption that each  $X_n$  is closed in  $X_{n+1}$ ?

(d) Suppose that  $X$  and  $X_n$  are as in Exercise 8.10, and that  $X_n \neq X_{n+1}$  for each  $n$ . Recall (see the lecture) that  $X$  is nonmetrizable. Does this result hold without the assumption that each  $X_n$  is closed in  $X_{n+1}$ ?