Inductive topologies. Direct sums

(exercises for Lecture 6)

7.1. Let X be a vector space, $(X_i)_{i\in I}$ a family of topological vector spaces, and $(\varphi_i\colon X_i \to X)_{i\in I}$ a family of linear maps. Let $\tau_{\text{ind.vec}}$ (resp., $\tau_{\text{ind.lc}}$) denote the inductive vector topology (resp., the inductive locally convex topology) on X generated by $(\varphi_i)_{i\in I}$.
(a) Show that $\tau_{\text{ind.lc}}$ is a unique locally convex topology on X having the following property: if Y is a locally convex space, then a linear map $\psi\colon X \to Y$ is continuous if and only if all the maps $\psi\circ\varphi_i\colon X_i \to Y$ are continuous.
(b) Prove a similar characteristic property of $\tau_{\text{ind.vec}}$.
(c) Show that the family of all absorbing, absolutely convex sets $U \subset X$ such that, for each $i \in I$, $\varphi^{-1}(U_i)$ is a neighborhood of zero in X_i, is a base of neighborhoods of zero in $(X, \tau_{\text{ind.lc}})$.
(d) Give a similar description of a base of neighborhoods of zero in $(X, \tau_{\text{ind.vec}})$.

7.2. Let X be a topological vector space, and let $\tau_{\text{ind.lc}}$ denote the inductive locally convex topology on X generated by the identity map $\mathbf{1}_X$. State and prove a universal property of $X_{\text{lc}} = (X, \tau_{\text{ind.lc}})$. Describe a defining family of seminorms for X_{lc}. Find X_{lc} in the case where X is **(a)** $L^p[0,1]$, $0 \leqslant p < 1$; **(b)** ℓ^p, $0 < p < 1$.

7.3. Let X, X_i, φ_i be as in Exercise 7.1. Assume also that all X_i are locally convex, and that $X = \sum_{i\in I}\varphi_i(X_i)$.
(a)* Prove that, if I is at most countable, then $\tau_{\text{ind.lc}} = \tau_{\text{ind.vec}}$. (*Hint*: it suffices to show that every neighborhood of 0 in $(X, \tau_{\text{ind.vec}})$ contains a set of the form $\sum_i \varphi_i(U_i)$, where $U_i \subset X_i$ are neighborhoods of 0.)
(b)* Construct an example such that I is uncountable and $\tau_{\text{ind.lc}} \neq \tau_{\text{ind.vec}}$. (*Hint*: consider the strongest locally convex topology on a vector space of uncountable dimension.)
(c) Construct an example such that I is finite and $\tau_{\text{ind.vec}} \neq \tau_{\text{ind}}$, were τ_{ind} is the strongest topology on X making all φ_i continuous.

7.4. **(a)** Show that the locally convex direct sum of a family of locally convex spaces is their coproduct in LCS (in the category-theoretic sense).
(b) Show that an infinite family of nonzero normed spaces does not have a coproduct in the category of normed spaces and continuous linear maps.

7.5. Let $(X_i)_{i\in I}$ be a family of locally convex spaces. Prove that
(a) if I is finite, then $\bigoplus_{i\in I} X_i = \prod_{i\in I} X_i$ as topological vector spaces;
(b) if I is infinite, and if the topology on X_i is nontrivial for all $i \in I$, then the standard (inductive) topology on $\bigoplus_{i\in I} X_i$ is strictly stronger than the topology induced from $\prod_{i\in I} X_i$.

7.6. Let $(X_i)_{i\in I}$ be a family of nonzero locally convex spaces. Show that
(a) $\bigoplus_{i\in I} X_i$ is Hausdorff $\iff$ all the X_i's are Hausdorff;
(b) $\bigoplus_{i\in I} X_i$ is normable $\iff$ all the X_i's are normable, and I is finite;
(c) $\bigoplus_{i\in I} X_i$ is metrizable $\iff$ all the X_i's are metrizable, and I is finite.

As a corollary (see Exercise 3.8), an infinite-dimensional strongest locally convex space is not metrizable.

7.7. Let $(X_i)_{i\in I}$ be a family of Hausdorff locally convex spaces. Show that a set $B \subset \bigoplus_{i\in I} X_i$ is bounded if and only if there exists a finite subset $J \subset I$ such that $B \subset \prod_{j\in J} B_j$, where $B_j \subset X_j$ are bounded sets.

7.8. Let X be a locally compact, second countable Hausdorff topological space, and let $C_c(X)$ be the space of compactly supported continuous functions on X topologized in the standard way. Let $C(X)_{\geqslant 0}$ denote the set of all nonnegative continuous functions on X. Given $a \in C(X)_{\geqslant 0}$, define a seminorm $\|\cdot\|_a$ on $C_c(X)$ by letting $\|f\|_a = \sup_{x\in X} |f(x)|a(x)$. Show that the family $\{\|\cdot\|_a : a \in C(X)_{\geqslant 0}\}$ of seminorms is defining for $C_c(X)$.

7.9. Let $U \subset \mathbb{R}^n$ be an open set, and let $C_c^\infty(U)$ be the space of compactly supported smooth functions on U topologized in the standard way. Let $\mathscr{V}$ denote the set of all tuples of the form $v = (v_\alpha)_{\alpha\in\mathbb{Z}^n_{\geqslant 0}}$, where $v_\alpha \in C(U)_{\geqslant 0}$ and the family $(\operatorname{supp} v_\alpha)_{\alpha\in\mathbb{Z}^n_{\geqslant 0}}$ is locally finite[1]. For each $v = (v_\alpha) \in \mathscr{V}$ define a seminorm $\|\cdot\|_v$ on $C_c^\infty(U)$ by letting

$$\|f\|_v = \sup_{\alpha\in\mathbb{Z}^n_{\geqslant 0}} \sup_{x\in U} |D^\alpha f(x)| v_\alpha(x).$$

Show that the family $\{\|\cdot\|_v : v \in \mathscr{V}\}$ of seminorms is defining for $C_c^\infty(U)$.

[1] A family $(X_i)_{i\in I}$ of subsets of a topological space X is *locally finite* if each $x \in X$ has a neighborhood U such that $U \cap X_i = \varnothing$ for all but finitely many $i \in I$.