Projective topologies. Products and projective limits

(EXERCISES FOR LECTURE 5)

6.1. Let X be a vector space, $(X_i)_{i\in I}$ be a family of topological vector spaces, and $(\varphi_i\colon X\to X_i)_{i\in I}$ be a family of linear maps. Equip X with the projective topology generated by (φ_i). Show that
(a) X is a topological vector space;
(b) a set $B\subset X$ is bounded if and only if $\varphi_i(B)$ is bounded in X_i for all $i\in I$;
(c) if each X_i is locally convex, then so is X;
(d) if, for each $i\in I$, P_i is a defining family of seminorms on X_i, then $\{p\circ\varphi_i : i\in I,\ p\in P_i\}$ is a defining family of seminorms on X.

6.2. In the setting of Exercise 6.1, define $\varphi\colon X\to\prod_i X_i$ by $\varphi(x)_i=\varphi_i(x)$. Show that the projective topology on X generated by the family $(\varphi_i)_{i\in I}$ is identical to the projective topology generated by $\{\varphi\}$. In particular, if $\bigcap_i \operatorname{Ker}\varphi_i=0$, then X is topologically isomorphic to a subspace of $\prod_i X_i$.

6.3. **(a)** Show that the product of a family of topological vector spaces is their product in TVS (in the category-theoretic sense).
(b) Show that an infinite family of nonzero normed spaces does not have a product in the category of normed spaces and continuous linear maps.

6.4. Let $(X_i)_{i\in I}$ be a family of nonzero locally convex spaces. Show that
(a) $\prod_{i\in I} X_i$ is Hausdorff $\iff$ all the X_i's are Hausdorff;
(b) $\prod_{i\in I} X_i$ is normable $\iff$ all the X_i's are normable, and I is finite;
(c) $\prod_{i\in I} X_i$ is metrizable $\iff$ all the X_i's are metrizable, and I is at most countable.

6.5. Let $F=(X_i,\varphi_{ij})$ be a projective system of topological vector spaces indexed by a directed set I. Let

$$X=\Big\{x=(x_i)\in\prod_{i\in I}X_i : x_i=\varphi_{ij}(x_j)\ \forall i<j\Big\}.$$

Equip X with the projective topology generated by $(\varphi_i)_{i\in I}$ (or, equivalently, with the topology induced from $\prod_{i\in I}X_i$, see Exercise 6.2). For each $i\in I$, let $\varphi_i\colon X\to X_i$ denote the restriction to X of the canonical projection onto the ith factor. Show that
(a) (X,φ_i) is the (category-theoretic) projective limit of F in TVS;
(b) if all the X_i's are Hausdorff, then X is closed in $\prod_{i\in I}X_i$;
(c) if, for each $i\in I$, β_i is a base of neighborhoods of 0 in X_i, then $\{\varphi_i^{-1}(U) : i\in I,\ U\in\beta_i\}$ is a base of neighborhoods of 0 in X.

6.6. Let (X_i,φ_{ij}) be a projective system of topological vector spaces, $X=\varprojlim(X_i,\varphi_{ij})$, and let Y be a vector subspace of X. For each i, let $Y_i=\varphi_i(Y)\subset X_i$ (where $\varphi_i\colon X\to X_i$ is the canonical map).
(a) Show that $\overline{Y}=\varprojlim(\overline{Y_i},\varphi_{ij}|_{\overline{Y_j}})$. In particular, if Y_i is dense in X_i for all i, then Y is dense in X.
(b) Do we always have $Y=\varprojlim(Y_i,\varphi_{ij}|_{Y_j})$?

6.7. Let $(X_i)_{i\in I}$ be a family of topological vector spaces. Construct a topological isomorphism $\prod_{i\in I}X_i\cong\varprojlim\{\prod_{j\in J}X_j : J\subset I \text{ is a finite subset}\}$.

6.8. Let X be a locally compact Hausdorff topological space.
(a) Construct a topological isomorphism $C(X)\cong\varprojlim\{C(K) : K\subset X \text{ is a compact set}\}$.
(b) Assume that X is second countable, and let $(K_j)_{j\in\mathbb{N}}$ be a compact exhaustion of X, i.e., a sequence of compact sets such that $X=\bigcup K_j$ and such that $K_j\subset\operatorname{Int}K_{j+1}$ for all j. (A subexercise: prove that a compact exhaustion exists.) Construct a topological isomorphism $C(X)\cong\varprojlim_{j\in\mathbb{N}}C(K_j)$.

6.9. Let $U \subset \mathbb{C}$ be an open set, and let $(K_j)_{j\in\mathbb{N}}$ be a compact exhaustion of U. For each j, let $\mathscr{A}(K_j)$ denote the subspace of $C(K_j)$ consisting of functions holomorphic on $\operatorname{Int} K_j$. Construct a topological isomorphism $\mathscr{O}(U) \cong \varprojlim_{j\in\mathbb{N}} \mathscr{A}(K_j)$.

6.10. Let $U \subset \mathbb{C}$ be an open set. Represent $\mathscr{O}(U)$ as the projective limit of a sequence of Hilbert spaces.

6.11. Define $\varphi\colon \ell^\infty \to \ell^\infty$ by $\varphi(x_1, x_2, \ldots) = (x_1, x_2/2, x_3/3, \ldots)$.
(a) Show that the projective limit of the sequence $\ell^\infty \xleftarrow{\varphi} \ell^\infty \xleftarrow{\varphi} \ell^\infty \xleftarrow{\varphi} \cdots$ is topologically isomorphic to s, the space of rapidly decreasing sequences (see Exercise sheet 2).
(b) Show that replacing ℓ^∞ by ℓ^p (where $1 \leqslant p < \infty$) or by c_0 yields the same projective limit.

6.12. Construct a topological isomorphism $C^\infty(\mathbb{R}) \cong \varprojlim_{k\in\mathbb{N}} C^k[-k,k]$.

6.13*. Represent $C^\infty(\mathbb{R})$ as the projective limit of a sequence of Hilbert spaces.

6.14. Represent the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ as the projective limit of a sequence of **(a)** Banach spaces; **(b)*** Hilbert spaces.