Quotients, kernels, cokernels

(EXERCISES FOR LECTURE 4)

Notation. Let TVS denote the category of topological vector spaces and continuous linear maps, and let HTVS denote the full subcategory of TVS consisting of Hausdorff spaces.

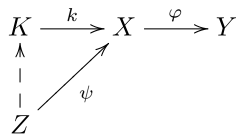
5.1. Let X be a topological vector space, let $Y \subset X$ be a vector subspace, and let $\pi\colon X \to X/Y$ denote the quotient map. Show that
(a) if β is a base of neighborhoods of 0 in X, then $\{\pi(U) : U \in \beta\}$ is a base of neighborhoods of 0 in X/Y;
(b) X is Hausdorff if and only if $\{0\}$ is closed in X;
(c) X/Y is Hausdorff if and only if Y is closed in X.

5.2. Let X be a locally convex space, P be a directed defining family of seminorms on X, and Y be a vector subspace of X.
(a) Show that, for each $p \in P$, we have $\pi(U_p) = U_{\hat{p}}$, where $\pi\colon X \to X/Y$ is the quotient map and $\hat{p}$ is the quotient seminorm of p.
(b) Show that $\hat{P} = \{\hat{p} : p \in P\}$ is a defining family of seminorms on X/Y.
(c) Does **(b)** hold if P is not assumed to be directed?

5.3. Let p be a seminorm on a vector space X, let Y be a vector subspace of X such that $Y \subset p^{-1}(0)$, and let $\hat{p}$ denote the quotient seminorm on X/Y. Show that $\hat{p}(x+Y) = p(x)$ for all $x \in X$.

5.4. Show that the inclusion functor HTVS $\hookrightarrow$ TVS has a left adjoint, and describe it explicitly. (*Hint:* consider $X_h = X/\overline{\{0\}}$.)

Definition 5.1. Let $\mathscr{A}$ be a category having a zero object, and let $\varphi\colon X \to Y$ be a morphism in $\mathscr{A}$. A *kernel* of φ is a pair (K, k), where $K \in \mathscr{A}$ and $k\colon K \to X$, such that
(i) $\varphi \circ k = 0$;
(ii) for each morphism $\psi\colon Z \to X$ in $\mathscr{A}$ satisfying $\varphi \circ \psi = 0$ there exists a unique morphism $Z \to K$ making the following diagram commute:



We write $K = \operatorname{Ker}\varphi$ and $k = \ker\varphi$. Equivalently, a kernel of φ is an object $\operatorname{Ker}\varphi$ together with a natural isomorphism

$$\operatorname{Hom}(Z, \operatorname{Ker}\varphi) \cong \operatorname{Ker}\big(\operatorname{Hom}(Z,X) \to \operatorname{Hom}(Z,Y)\big).$$

A *cokernel* of φ is a pair (C, c), where $C \in \mathscr{A}$ and $c\colon Y \to C$, such that (C, c) is a kernel of φ in the dual category $\mathscr{A}^{\mathrm{op}}$ (draw the respective diagram!). We write $C = \operatorname{Coker}\varphi$ and $c = \operatorname{coker}\varphi$. Equivalently, a cokernel of φ is an object $\operatorname{Coker}\varphi$ together with a natural isomorphism

$$\operatorname{Hom}(\operatorname{Coker}\varphi, Z) \cong \operatorname{Ker}\big(\operatorname{Hom}(Y,Z) \to \operatorname{Hom}(X,Z)\big).$$

5.5. **(a)** Show that the kernel of a morphism $\varphi\colon X \to Y$ in TVS is the subspace $\varphi^{-1}(0)$, and that the cokernel of φ is the quotient $Y/\varphi(X)$.
(b) Describe kernels and cokernels of morphisms in HTVS.

Definition 5.2. Let $\mathscr{A}$ be a category having a zero object. A morphism $\varphi\colon X \to Y$ in $\mathscr{A}$ is a *kernel* (resp., a *cokernel*) if there exists a morphism $\psi\colon Y \to Z$ (resp., $\psi\colon Z \to X$) such that $\varphi = \ker\psi$ (resp., $\varphi = \operatorname{coker}\psi$).

5.6. (a) Show that a morphism φ in TVS is a kernel if and only if it is topologically injective, and that φ is a cokernel if and only if it is open.
(b) Obtain a similar characterization of kernels and cokernels in HTVS.

Definition 5.3. Let $\mathscr{A}$ be a category having a zero object. Suppose that each morphism in $\mathscr{A}$ has a kernel and a cokernel. We define the *image* $(\operatorname{Im}\varphi, \operatorname{im}\varphi)$ of a morphism φ in $\mathscr{A}$ to be the kernel of the cokernel of φ, and the *coimage* $(\operatorname{Coim}\varphi, \operatorname{coim}\varphi)$ of φ to be the cokernel of the kernel of φ. Thus for each $\varphi\colon X \to Y$ there is a unique $\bar\varphi\colon \operatorname{Coim}\varphi \to \operatorname{Im}\varphi$ making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & Y \\ {\scriptstyle \operatorname{coim}\varphi}\downarrow & & \uparrow{\scriptstyle \operatorname{im}\varphi} \\ \operatorname{Coim}\varphi & \underset{\bar\varphi}{\dashrightarrow} & \operatorname{Im}\varphi \end{array}$$

We say that φ is *strict* if $\bar\varphi$ is an isomorphism.

5.7. (a) Describe the image and the coimage of each morphism in the categories TVS and HTVS.
(b) Show that a morphism $\varphi\colon X \to Y$ in TVS is strict if and only if φ is an open map of X onto $\varphi(X)$.
(c) Describe strict morphisms in HTVS.