Bounded sets

(exercises for Lecture 4)

4.1. Let X be a topological vector space. Show that
(a) if $B_1, B_2 \subset X$ are bounded, then $B_1 + B_2$ is bounded;
(b) if $S \subset \mathbb{K}$ and $B \subset X$ are bounded, then $S \cdot B$ is bounded.

4.2. Show that a relatively compact subset of a topological vector space is bounded.

4.3. Let X be a Hausdorff topological vector space. Assume that X has a bounded neighborhood of 0. Does this imply that X is normable? (For locally convex spaces, the answer is yes by Kolmogorov's criterion, see the lectures.)

4.4. Construct a linear map between locally convex spaces X and Y which takes bounded subsets of X to bounded subsets of Y, but which is not sequentially continuous. Using this, find an example of a nonbornological locally convex space. (*Hint:* weak topologies.)

4.5*. Construct a discontinuous, sequentially continuous linear map between locally convex spaces.

Definition 4.1. A locally convex space X is *semi-Montel* if each bounded subset of X is relatively compact.

4.6. Let S be a set. **(a)** Show that $B \subset \mathbb{K}^S$ is bounded if and only if for each $s \in S$ the set $\{f(s) : f \in B\}$ is bounded in $\mathbb{K}$. **(b)** Deduce that $\mathbb{K}^S$ is semi-Montel.

4.7. Let X be a vector space equipped with the strongest locally convex topology. **(a)** Show that $B \subset X$ is bounded if and only if B is contained in a finite-dimensional vector subspace $Y \subset X$ and is bounded for the standard topology on Y (i.e., for the topology generated by any norm on Y). **(b)** Deduce that X is semi-Montel.

4.8. **(a)** Show that $B \subset C(\mathbb{R})$ is bounded if and only if there exists a nonnegative $g \in C(\mathbb{R})$ such that for all $t \in \mathbb{R}$ and all $f \in B$ we have $|f(t)| \leqslant g(t)$. **(b)** Is $C(\mathbb{R})$ semi-Montel?

4.9. **(a)** Show that $B \subset s$ is bounded if and only if there exists $y \in s$ such that for all n and all $x \in B$ we have $|x_n| \leqslant |y_n|$. **(b)** Deduce that s is semi-Montel.

Hint to **(b)**: embed s into the product of countably many copies of c_0 (the Banach space of sequences vanishing at infinity) and use a standard compactness criterion in c_0.

4.10. Given $f \in \mathscr{O}(\mathbb{C})$, let $(c_n(f))_{n \geqslant 0}$ denote the sequence of the Taylor coefficients of f at 0.
(a) Show that $B \subset \mathscr{O}(\mathbb{C})$ is bounded if and only if there exists $g \in \mathscr{O}(\mathbb{C})$ such that for all n and all $f \in B$ we have $|c_n(f)| \leqslant |c_n(g)|$. **(b)** Deduce that $\mathscr{O}(\mathbb{C})$ is semi-Montel.

Hint to **(b)**: see the hint to the previous exercise.

4.11. **(a)** Show that $B \subset C^\infty(\mathbb{R})$ is bounded if and only if there exists a sequence $(g_n)_{n \geqslant 0}$ of nonnegative functions in $C^\infty(\mathbb{R})$ such that for all $t \in \mathbb{R}$, all $n \in \mathbb{Z}_{\geqslant 0}$, and all $f \in B$ we have $|f^{(n)}(t)| \leqslant g_n(t)$. **(b)** Deduce that $C^\infty(\mathbb{R})$ is semi-Montel.

Hint to **(b)**: embed $C^\infty(\mathbb{R})$ into the product of countably many copies of $C[a, b]$ (for suitable a, b) and use the Arzela-Ascoli theorem.