Normable and metrizable locally convex spaces

(EXERCISES FOR LECTURE 3)

3.1. Let S be an infinite set. Show that there are no continuous norms on $\mathbb{K}^S$. As a corollary, $\mathbb{K}^S$ is not normable.

3.2. Let X be a noncompact, completely regular (i.e., Tychonoff) topological space. Show that there are no continuous norms on $C(X)$. As a corollary, $C(X)$ is not normable.

3.3. Let $U \subset \mathbb{R}^n$ be a nonempty open set. Show that there are no continuous norms on $C^\infty(U)$. As a corollary, $C^\infty(U)$ is not normable.

3.4. Let $U \subset \mathbb{C}^n$ be a nonempty open set. When does $\mathscr{O}(U)$ have a continuous norm?

3.5. Show that the following spaces are not normable, although each of them has a continuous norm:
(a) s; **(b)** $C^\infty[a, b]$; **(c)** $\mathscr{S}(\mathbb{R}^n)$; **(d)** $\mathscr{O}(U)$ (where U is a nonempty domain in $\mathbb{C}$).

3.6. Prove that the following spaces are metrizable:
(a) $C(X)$, where X is a second countable, locally compact topological space;
(b) all spaces from Exercises 3.3–3.5.

3.7. Let S be a set. Show that $\mathbb{K}^S$ is metrizable if and only if S is at most countable.

3.8. Show that a vector space equipped with the strongest locally convex topology is metrizable if and only if it is finite-dimensional.

3.9. Let X be a normed space. Show that
(a) the dual space X' equipped with the weak* topology is metrizable if and only if the dimension of X is at most countable;
(b) X equipped with the weak topology is metrizable if and only if it is finite-dimensional.

3.10*. Let X be a finite-dimensional vector space. Show that there is only one topology on X which makes X into a Hausdorff topological vector space, and that this topology is determined by any norm on X. (This result was proved at the lectures in the special case of locally convex topologies.)

3.11*. Prove that a Hausdorff topological vector space is finite-dimensional if and only if it is locally compact.

3.12*. Prove that a topological vector space is semimetrizable if and only if its topology is generated by an F-seminorm. (This result was proved at the lectures in the special case of locally convex spaces.)