Continuous linear operators. Equivalent families of seminorms

(EXERCISES FOR LECTURES 2–3)

Recall (see the lectures) that the space s of *rapidly decreasing sequences* is defined by

$$s = \Big\{ x = (x_n) \in \mathbb{K}^{\mathbb{N}} : \|x\|_k = \sup_{n\in\mathbb{N}} |x_n| n^k < \infty \ \forall k \in \mathbb{Z}_{\geqslant 0} \Big\}. \tag{1}$$

The topology on s is given by the seminorms $\|\cdot\|_k$ ($k \in \mathbb{Z}_{\geqslant 0}$). Similarly, one defines the space $s(\mathbb{Z})$ of rapidly decreasing sequences on $\mathbb{Z}$ (more exactly, we replace $\mathbb{N}$ by $\mathbb{Z}$ and n^k by $(1+|n|)^k$ in (1)).

2.1. Let $\lambda = (\lambda_n) \in \mathbb{K}^{\mathbb{N}}$. Consider the *diagonal operator*

$$M_\lambda \colon \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}, \quad (x_1, x_2, \ldots) \mapsto (\lambda_1 x_1, \lambda_2 x_2, \ldots).$$

(a) Show that M_λ is continuous.
(b) Find a condition on λ that is equivalent to $M_\lambda(s) \subset s$.
(c) Find a condition on λ that is necessary and sufficient for M_λ to be a continuous map of s to s.

2.2. Describe all continuous linear functionals on the spaces **(a)** $\mathbb{K}^{\mathbb{N}}$; **(b)** s.

Recall (see the lectures) that the *Schwartz space* $\mathscr{S}(\mathbb{R}^n)$ is defined by

$$\mathscr{S}(\mathbb{R}^n) = \Big\{ f \in C^\infty(\mathbb{R}^n) : \|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty \ \forall \alpha, \beta \in \mathbb{Z}^n_{\geqslant 0} \Big\}.$$

The topology on $\mathscr{S}(\mathbb{R}^n)$ is given by the seminorms $\|\cdot\|_{\alpha,\beta}$ ($\alpha, \beta \in \mathbb{Z}^n_{\geqslant 0}$).

2.3. (a) Let $U \subset \mathbb{R}^n$ be an open set. Consider a differential operator

$$D = \sum_{|\alpha| \leqslant N} a_\alpha D^\alpha, \tag{2}$$

where $a_\alpha \in C^\infty(U)$. Show that D is a continuous operator on $C^\infty(U)$.
(b) Find a reasonable condition on $a_\alpha \in C^\infty(\mathbb{R}^n)$ that is sufficient for D to be a continuous operator on $\mathscr{S}(\mathbb{R}^n)$.
(c) Let us equip the space $\mathbb{K}[[x_1, \ldots, x_n]]$ of formal power series with the topology of convergence of each coefficient (in other words, we identify $\mathbb{K}[[x_1, \ldots, x_n]]$ with $\mathbb{K}^{\mathbb{Z}^n_{\geqslant 0}}$ equipped with the product topology). Show that for each $a_\alpha \in \mathbb{K}[[x_1, \ldots, x_n]]$ formula (2) defines a continuous operator on $\mathbb{K}[[x_1, \ldots, x_n]]$.
(d) Let $U \subset \mathbb{C}$ be an open set, and let $a_1, \ldots, a_N \in \mathscr{O}(U)$. Show that the differential operator

$$\sum_{k=0}^{N} a_k \frac{d^k}{dz^k}$$

is continuous on $\mathscr{O}(U)$.

2.4. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and let μ denote the normalized length measure on $\mathbb{T}$ ("normalized" means that the measure of $\mathbb{T}$ is 1). Show that the *Fourier transform*

$$\mathscr{F} \colon C^\infty(\mathbb{T}) \to s(\mathbb{Z}), \quad (\mathscr{F} f)(n) = \int_{\mathbb{T}} f(z) z^{-n} \, d\mu(z),$$

is a topological isomorphism of $C^\infty(\mathbb{T})$ onto $s(\mathbb{Z})$.

2.5. Show that an open linear operator between topological vector spaces is surjective.

2.6. Characterize **(a)** topologically injective and **(b)** open linear operators between locally convex spaces in terms of defining families of seminorms (in the spirit of the continuity criterion, see the lectures).

2.7. Show that the following families of seminorms on s are equivalent (where $1 \leqslant p < +\infty$):

(1) $\|x\|_k^{(\infty)} = \sup_n |x_n| n^k \quad (k \in \mathbb{Z}_{\geqslant 0})$;

(2) $\|x\|_k^{(1)} = \sum_n |x_n| n^k \quad (k \in \mathbb{Z}_{\geqslant 0})$;

(3) $\|x\|_k^{(p)} = \Big(\sum_n |x_n|^p n^{kp}\Big)^{1/p} \quad (k \in \mathbb{Z}_{\geqslant 0})$.

2.8. Show that the following families of seminorms on $\mathscr{S}(\mathbb{R}^n)$ are equivalent (where $1 \leqslant p < +\infty$):

(1) $\|f\|_{\alpha,\beta} = \sup\limits_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| \quad (\alpha, \beta \in \mathbb{Z}^n_{\geqslant 0})$;

(2) $\|f\|_{k,\beta} = \sup\limits_{x \in \mathbb{R}^n} \|x\|^k |D^\beta f(x)| \quad (k \in \mathbb{Z}_{\geqslant 0},\ ,\beta \in \mathbb{Z}^n_{\geqslant 0})$;

(3) $\|f\|_{k,\beta}^{(0)} = \sup\limits_{x \in \mathbb{R}^n} (1 + \|x\|)^k |D^\beta f(x)| \quad (k \in \mathbb{Z}_{\geqslant 0},\ ,\beta \in \mathbb{Z}^n_{\geqslant 0})$;

(4) $\|f\|_{k,\beta}^{(1)} = \int_{\mathbb{R}^n} (1 + \|x\|)^k |D^\beta f(x)|\, dx \quad (k \in \mathbb{Z}_{\geqslant 0},\ ,\beta \in \mathbb{Z}^n_{\geqslant 0})$;

(5) $\|f\|_{k,\beta}^{(p)} = \Big(\int_{\mathbb{R}^n} (1 + \|x\|)^{kp} |D^\beta f(x)|^p\, dx\Big)^{1/p} \quad (k \in \mathbb{Z}_{\geqslant 0},\ ,\beta \in \mathbb{Z}^n_{\geqslant 0})$.

2.9. Let U be a domain in $\mathbb{C}$, and let $\mathscr{O}(U)$ denote the space of holomorphic functions on U. Choose a compact exhaustion $\{U_i\}_{i \in \mathbb{N}}$ of U (i.e., $U = \bigcup_i U_i$, U_i is open, $\overline{U_i}$ is compact, and $\overline{U_i} \subset U_{i+1}$ for all $i \in \mathbb{N}$). Let $p \in [1, +\infty)$, and let μ denote the Lebesgue measure on $\mathbb{C}$. Show that the following families of seminorms on $\mathscr{O}(U)$ are equivalent:

(1) $\|f\|_K = \sup\limits_{z \in K} |f(z)| \quad (K \subset U$ is a compact set$)$;

(2) $\|f\|_{k,\ell,K} = \sup\limits_{z = x+iy \in K} \left| \dfrac{\partial^{k+\ell} f(z)}{\partial x^k\, \partial y^\ell} \right| \quad (K \subset U$ is a compact set, $k, \ell \in \mathbb{Z}_{\geqslant 0})$;

(3) $\|f\|_i^{(1)} = \int_{U_i} |f(z)|\, d\mu(z) \quad (i \in \mathbb{N})$;

(4) $\|f\|_i^{(p)} = \Big(\int_{U_i} |f(z)|^p\, d\mu(z)\Big)^{1/p} \quad (i \in \mathbb{N})$.

Remark. The equivalence of (1) and (2) in Exercise 2.9 means that the topology of compact convergence and the topology induced from $C^\infty(U)$ are the same on $\mathscr{O}(U)$.

2.10. Let $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$. Given $f \in \mathscr{O}(\mathbb{D}_R)$, let $c_n(f) = f^{(n)}(0)/n!$. Choose $p \in [1, +\infty)$, and let μ denote the Lebesgue measure on the circle $|z| = r$. Show that the following families of seminorms on $\mathscr{O}(\mathbb{D}_R)$ are equivalent:

(1) $\|f\|_K = \sup\limits_{z \in K} |f(z)| \quad (K \subset \mathbb{D}_R$ is a compact set$)$;

(2) $\|f\|_r^{(1)} = \sum_{n=0}^\infty |c_n(f)| r^n \quad (0 < r < R)$;

(3) $\|f\|_r^{(p)} = \Big(\sum_{n=0}^\infty |c_n(f)|^p\, r^{np}\Big)^{1/p} \quad (0 < r < R)$;

(4) $\|f\|_r^{\infty} = \sup\limits_{n \geqslant 0} |c_n(f)| r^n \quad (0 < r < R)$;

(5) $\|f\|_r^{I} = \int_{|z|=r} |f(z)|\, d\mu(z) \quad (0 < r < R)$;

(6) $\|f\|_r^{I,p} = \Big(\int_{|z|=r} |f(z)|^p\, d\mu(z)\Big)^{1/p} \quad (0 < r < R)$.