Nuclear operators and nuclear spaces

(EXERCISES FOR LECTURES 13–15)

11.1. Let $S\colon X \to Y$ and $T\colon Y \to Z$ be bounded linear operators between Banach spaces. Show that, if either S or T is nuclear, then so is TS.

11.2. Let $X = \ell^p$ (where $1 \leqslant p \leqslant \infty$) or $X = c_0$. Given $\alpha = (\alpha_n) \in \ell^\infty$, consider the diagonal operator

$$M_\alpha\colon X \to X, \qquad (x_1, x_2, \ldots) \mapsto (\alpha_1 x_1, \alpha_2 x_2, \ldots).$$

Show that M_α is nuclear iff $\alpha \in \ell^1$, and that $\|M_\alpha\|_{\mathcal{N}} = \|\alpha\|_1$.

(*Hint:* for $1 < p < \infty$, use Exercise 10.2 (a)).

11.3. Let $I = C[a,b]$ and let $K \in C(I \times I)$. Prove that the integral operator

$$T\colon C(I) \to C(I), \qquad (Tf)(x) = \int_a^b K(x,y) f(y)\, dy$$

is nuclear.

11.4. Let T be a bounded linear operator on ℓ^1 with matrix (a_{ij}) in the standard basis (e_j) (i.e., $Te_j = \sum_i a_{ij} e_i$ for all $j \in \mathbb{N}$). Find a nuclearity criterion for T and calculate the nuclear norm of T in terms of the matrix elements a_{ij}.

11.5. Do the same as in the previous exercise with c_0 in place of ℓ^1.

11.6. Let Y be a Banach space and $Z \subset Y$ a closed vector subspace. Suppose that there exists a Banach space X such that the map $Z \mathbin{\widehat{\otimes}_\pi} X' \to Y \mathbin{\widehat{\otimes}_\pi} X'$ generated by the inclusion $Z \hookrightarrow Y$ is not topologically injective (cf. Exercises 10.6–10.8). Assume also that the canonical map $Y \mathbin{\widehat{\otimes}_\pi} X' \to \mathscr{L}(X,Y)$ is injective. Prove that there exists a nuclear operator from X to Y whose range is contained in Z but which is not a nuclear operator from X to Z.

11.7. Let M be a complex manifold (for simplicity, you may assume that M is an open subset of $\mathbb{C}$), and let U be an open, relatively compact subset of M. Show that the restriction map $\mathscr{O}(M) \to \mathscr{O}(U)$ is nuclear.

11.8. Let X and Y be vector spaces, $\varphi\colon X \to Y$ a linear map, and $B \subset X$ a Banach disk. Show that $\varphi(B)$ is a Banach disk in Y.

11.9. Let X be a locally convex space, and let $B \subset X$ be an absolutely convex bounded set. Show that

(a) the inclusion $j_B\colon X_B \to X$ is continuous;

(b) if X is Hausdorff, then X_B is a normed space;

(c) if X is Hausdorff and B is complete, then B is a Banach disk.

11.10. Let X be a locally convex space, and let p be a continuous seminorm on X. Consider the set $B_p = \{f \in X' : |f(x)| \leqslant 1 \ \forall x \in U_p\}$. Show that B_p is a Banach disk, and construct an isometric isomorphism $(X')_{B_p} \cong (X_p)'$.

11.11. Let φ_1 and φ_2 be nuclear operators between locally convex spaces. Show that the operators $\varphi_1 \otimes_\pi \varphi_2$ and $\varphi_1 \mathbin{\widehat{\otimes}_\pi} \varphi_2$ are nuclear.

11.12. Let $\varphi\colon X \to Y$ be a nuclear operator between locally convex spaces. Show that the dual operator $\varphi'\colon Y'_\beta \to X'_\beta$ is nuclear.

11.13. Prove that a complete locally convex space X is nuclear iff it can be represented as $X \cong \varprojlim(X_i, \varphi_{ij})$, where X_i are Banach spaces and the connecting maps φ_{ij} are nuclear for all $i < j$.

11.14. Let S^n be the n-sphere, and let $p \in S^n$. Construct a topological isomorphism

$$\mathscr{S}(\mathbb{R}^n) \cong \{f \in C^\infty(S^n) : Df(p) = 0 \quad \forall D \in A\},$$

where A is the algebra of linear operators on $C^\infty(S^n)$ generated (as a subalgebra of $\operatorname{End} C^\infty(S^n)$) by vector fields. As a corollary (see the lecture), $\mathscr{S}(\mathbb{R}^n)$ is nuclear.

11.15. Show that the strongest locally convex space of uncountable dimension (or, equivalently, the locally convex direct sum of uncountably many copies of $\mathbb{K}$) is not nuclear.

11.16 (*the Grothendieck-Pietsch criterion)***.** Prove that a Köthe space $\lambda^1(I, P)$ is nuclear iff for every $p \in P$ there exist $q \in P$ and $\lambda \in \ell^1(I)_{\geqslant 0}$ such that $p_i \leqslant \lambda_i q_i$ for all $i \in I$. (We proved this at the lecture under the assumption that $I = \mathbb{N}$ and $p_i > 0$ for all $p \in P$, $i \in I$.)

11.17. Let P be a countable Köthe set. Show that $\lambda^1(I, P)$ is nuclear iff $\lambda^1(I, P) = \lambda^\infty(I, P)$ (as vector subspaces of $\mathbb{K}^I$).

11.18. Show that each bounded subset of a complete nuclear locally convex space X is relatively compact in X.

11.19. Let X be a Fréchet space such that each bounded subset of X is relatively compact in X. Does this imply that X is nuclear?

11.20. Let $0 \to X_1 \to X_2 \to X_3 \to 0$ be an exact sequence of Fréchet spaces, and let Y be a Fréchet space. Suppose that X_1 is nuclear. Show that the sequence

$$0 \to X_1 \mathbin{\widehat{\otimes}_\pi} Y \to X_2 \mathbin{\widehat{\otimes}_\pi} Y \to X_2 \mathbin{\widehat{\otimes}_\pi} Y \to 0$$

is exact. (This was proved at the lecture under the assumption that either Y or X_2 is nuclear.)