Topological tensor products II

(exercises for Lectures 10–12)

10.1. Given $i \in \mathbb{N}$, let e_i denote the sequence with 1 in the ith slot and 0 elsewhere. Let $p \geqslant 1$.
(a) Show that, for each $n \in \mathbb{N}$ and each $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$, the following identities hold in $\ell^1 \otimes \ell^p$:

$$\Big\| \sum_{i=1}^n \lambda_i e_i \otimes e_i \Big\|_\pi = \sum_{i=1}^n |\lambda_i|, \quad \Big\| \sum_{i=1}^n \lambda_i e_i \otimes e_i \Big\|_\varepsilon = \Big(\sum_{i=1}^n |\lambda_i|^p \Big)^{1/p}.$$

(b) Show that, if $p > 1$, then the canonical map $\ell^1 \widehat{\otimes}_\pi \ell^p \to \ell^1 \widehat{\otimes}_\varepsilon \ell^p$ is neither topologically injective nor surjective.

10.2. Let $p, q > 1$ be such that $1/p + 1/q = 1$.
(a) Show that, for each $n \in \mathbb{N}$ and each $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$, the following identities hold in $\ell^p \otimes \ell^q$:

$$\Big\| \sum_{i=1}^n \lambda_i e_i \otimes e_i \Big\|_\pi = \sum_{i=1}^n |\lambda_i|, \quad \Big\| \sum_{i=1}^n \lambda_i e_i \otimes e_i \Big\|_\varepsilon = \max_{1 \leqslant i \leqslant n} |\lambda_i|.$$

(b) Show that the canonical map $\ell^p \widehat{\otimes}_\pi \ell^q \to \ell^p \widehat{\otimes}_\varepsilon \ell^q$ is neither topologically injective nor surjective.

Hint. To prove the nonsurjectivity in (b), it suffices to prove the injectivity of this map. For that purpose, show that $(p_n \otimes 1)(u) \to u$ for every $u \in \ell^p \widehat{\otimes}_\pi \ell^q$, where p_n is the projection onto the first n coordinates.

10.3. **(a)** Is $\ell^2 \widehat{\otimes}_\pi \ell^2$ topologically isomorphic to a Hilbert space? **(b)** What about $\ell^2 \widehat{\otimes}_\varepsilon \ell^2$?

10.4. Let $n \in \mathbb{N}$, and let $\zeta \in \mathbb{C}$ be an nth primitive root of unity. Define $u_n \in \ell^1 \otimes \ell^1$ by $u_n = \frac{1}{\sqrt{n}} \sum_{i,j=1}^n \zeta^{ij} e_i \otimes e_j$.
(a) Show that $\|u_n\|_\pi = n^{3/2}$ and $\|u_n\|_\varepsilon \leqslant n$.
(b) Show that the canonical map $\ell^1 \widehat{\otimes}_\pi \ell^1 \to \ell^1 \widehat{\otimes}_\varepsilon \ell^1$ is neither topologically injective nor surjective.
(*Hint for the 2nd estimate in* (a): u_n corresponds to a unitary operator on $\mathbb{C}^n$.)

10.5. Prove that the following canonical maps are neither topologically injective nor surjective:
(a) $c_0 \widehat{\otimes}_\pi c_0 \to c_0 \widehat{\otimes}_\varepsilon c_0$; **(b)** $C[0,1] \widehat{\otimes}_\pi C[0,1] \to C[0,1] \widehat{\otimes}_\varepsilon C[0,1]$.

Hint. For the topological noninjectivity, reduce (a) to the previous exercise and (b) to (a). For the nonsurjectivity in (a), see the hint to Exercise 10.2. For the nonsurjectivity in (b), use a similar argument, where p_n is the projection onto a suitable finite-dimensional subspace consisting of piecewise linear functions.

10.6 ($\otimes_\pi$ *does not respect subspaces*)**.** Let X be a Banach space and let $Y \subset X$ be a closed noncomplemented subspace having a predual Z (this means that Z' is topologically isomorphic to Y). Prove that the maps $Y \otimes_\pi Z \to X \otimes_\pi Z$ and $Y \widehat{\otimes}_\pi Z \to X \widehat{\otimes}_\pi Z$ induced by the inclusion $Y \hookrightarrow X$ are not topologically injective.

10.7. Given a Banach space X and a complemented subspace $Y \subset X$, let

$$\lambda(Y, X) = \inf\big\{ \|P\| : P \text{ is a continuous projection of } X \text{ onto } Y \big\}.$$

Suppose that for each $n \in \mathbb{N}$ we have a Banach space X_n and a finite-dimensional subspace $Y_n \subset X_n$ such that $\lambda(Y_n, X_n) \to \infty$ as $n \to \infty$. Let X denote the ℓ^1-sum of the X_n's, and let Y denote the ℓ^1-sum of the Y_n's. Show that X and Y satisfy the conditions of Exercise 10.6.

10.8. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ act on $C(\mathbb{T})$ via the regular representation, i.e., $(z \cdot f)(\zeta) = f(\zeta z)$ for all $f \in C(\mathbb{T})$, $z, \zeta \in \mathbb{T}$. Given $k \in \mathbb{Z}$, define $\chi_k \colon \mathbb{T} \to \mathbb{T}$, $\chi_k(z) = z^k$. For each $n \in \mathbb{N}$, we let $Y_n = \operatorname{span}\{\chi_k : -n \leqslant k \leqslant n\} \subset C(\mathbb{T})$. Suppose that P is a continuous projection of $C(\mathbb{T})$ onto Y_n. Define $Q \colon C(\mathbb{T}) \to C(\mathbb{T})$ by

$$Qf = \int_{\mathbb{T}} (z \cdot P \cdot z^{-1})(f)\, d\mu(z)$$

(the *average* of P), where μ is the normalized length measure on $\mathbb{T}$ (i.e., the usual length measure divided by 2π). Show that
(a) Q is a projection of $C(\mathbb{T})$ onto Y_n;
(b) $\|Q\| \leqslant \|P\|$;
(c) $Q(z \cdot f) = z \cdot Q(f)$ for all $z \in \mathbb{T}$ and $f \in C(\mathbb{T})$;
(d) there exists a unique projection $Q = Q_n$ of $C(\mathbb{T})$ onto Y_n satisfying (c), and Q_n takes each $f \in C(\mathbb{T})$ to the nth partial sum of its trigonometric Fourier series, i.e., $Q_n f = \sum_{k=-n}^{n} c_k(f)\chi_k$, where $c_k(f) = \int_{\mathbb{T}} f \bar{\chi}_k \, d\mu$;
(e) for each $f \in C(\mathbb{T})$ we have $Q_n f = D_n * f$, where $D_n = \sum_{k=-n}^{n} \chi_k$ is the nth Dirichlet kernel, and where the convolution $f * g$ of $f, g \in C(\mathbb{T})$ is defined by

$$(f * g)(z) = \int_{\mathbb{T}} f(\zeta) g(\zeta^{-1} z)\, d\mu(\zeta);$$

(f) $\lambda(Y_n, C(\mathbb{T})) = \|Q_n\| = \int_{\mathbb{T}} |D_n|\, d\mu \to \infty$ as $n \to \infty$.

Thus the spaces $X_n = C(\mathbb{T})$ and Y_n satisfy the conditions of Exercise 10.7.

10.9. Let X and Z be Banach spaces, and let $Y \subset X$ be a closed vector subspace. Suppose that the map $i \otimes \mathbf{1} \colon Y \otimes_\pi Z \to X \otimes_\pi Z$ induced by the inclusion map $i \colon Y \hookrightarrow X$ is not topologically injective (as in Exercise 10.6). Show that for each $n \in \mathbb{N}$ there exist finite-dimensional subspaces $X_n \subset X$, $Z_n \subset Z$ and an element $u \in Y_n \otimes_\pi Z_n$ (where $Y_n = Y \cap X_n$) such that $\|u\|_{X_n \otimes_\pi Z_n} \leqslant n^{-1} \|u\|_{Y_n \otimes_\pi Z_n}$.

10.10 ($\otimes_\varepsilon$ *does not respect quotients*)**.** Construct a surjective continuous linear map $X \to Y$ of Banach spaces and a Banach space Z such that the induced map $X \otimes_\varepsilon Z \to Y \otimes_\varepsilon Z$ is not open, and such that the induced map $X \mathbin{\widehat{\otimes}_\varepsilon} Z \to Y \mathbin{\widehat{\otimes}_\varepsilon} Z$ is not onto.