Topological tensor products

(EXERCISES FOR LECTURES 10–12)

9.1. Let X, Y, Z be topological vector spaces. Show that a bilinear map $X \times Y \to Z$ is continuous iff it is continuous at $(0,0)$.

9.2. Let X, Y, Z be locally convex spaces, and let P, Q, R be defining families of seminorms on X, Y, Z, resp. Suppose that P and Q are directed. Show that a bilinear map $\Phi\colon X \times Y \to Z$ is continuous iff for every $r \in R$ there exist $p \in P$, $q \in Q$, and $C > 0$ such that $r(\Phi(x,y)) \leqslant Cp(x)q(y)$ for all $x \in X$, $y \in Y$.

9.3. Let X, Y, Z be seminormed spaces. Show that $\mathscr{L}(X,Z)$ and $\mathscr{L}^{(2)}(X \times Y, Z)$ are normed spaces iff Z is a normed space.

9.4. Let X and Y be seminormed spaces. Show that the open unit ball of $X \otimes_\pi Y$ is the convex hull of the set $U_X \odot U_Y = \{x \otimes y : x \in U_X,\ y \in U_Y\}$, where U_X and U_Y are the open unit balls of X and Y, respectively. As a corollary, the projective tensor seminorm on $X \otimes Y$ is the Minkowski functional of $\operatorname{conv}(U_X \odot U_Y)$.

9.5. Let X and Y be seminormed spaces. Show that a seminorm α on $X \otimes Y$ is a reasonable cross-seminorm iff $\|\cdot\|_\varepsilon \leqslant \alpha \leqslant \|\cdot\|_\pi$.

9.6. Let X and Y be locally convex spaces. Show that
(a) the topology on $X \otimes_\pi Y$ is the strongest locally convex topology on $X \otimes Y$ making the canonical map $X \times Y \to X \otimes Y$, $(x,y) \mapsto x \otimes y$, continuous;
(b) if $\mathscr{U}$ and $\mathscr{V}$ are neighborhood bases at 0 in X and Y, respectively, then $\{\operatorname{conv}(U \odot V) : U \in \mathscr{U},\ V \in \mathscr{V}\}$ is a neighborhood base at 0 in $X \otimes_\pi Y$.

9.7. Let X and Y be infinite-dimensional normed spaces. Prove that the normed spaces $X \otimes_\pi Y$ and $X \otimes_\varepsilon Y$ are incomplete.

9.8. Formulate and prove **(a)** the commutativity and **(b)** the associativity of the tensor products $\otimes_\pi$, $\otimes_\varepsilon$, $\widehat{\otimes}_\pi$, $\widehat{\otimes}_\varepsilon$, and **(c)** their additivity in each variable.

9.9. Given seminormed spaces X, Y, Z, construct natural isometric isomorphisms $\mathscr{L}(X \otimes_\pi Y, Z) \cong \mathscr{L}(X, \mathscr{L}(Y,Z))$ and (assuming that Z is a Banach space) $\mathscr{L}(X \mathbin{\widehat{\otimes}_\pi} Y, Z) \cong \mathscr{L}(X, \mathscr{L}(Y,Z))$.

9.10. Given locally convex spaces X, Y, Z, construct a natural linear injection $\mathscr{L}(X \otimes_\pi Y, Z) \hookrightarrow \mathscr{L}(X, \mathscr{L}_b(Y,Z))$. Give an example showing that this map is not necessarily surjective. (*Hint:* take any infinite-dimensional normed space X, let $Z = \mathbb{K}$, and try to guess what Y is.)

9.11. Let X be a locally convex space, and $\{Y_i : i \in I\}$ be a family of locally convex spaces.
(a) Is the natural vector space isomorphism $X \otimes_\pi (\bigoplus_{i \in I} Y_i) \cong \bigoplus_{i \in I} (X \otimes_\pi Y_i)$ always a topological isomorphism? **(b)** The same question for $\otimes_\varepsilon$.

9.12. Given a normed space X, consider the normed spaces $X_1^n = (X^n, \|\cdot\|_1)$ and $X_\infty^n = (X^n, \|\cdot\|_\infty)$, where $\|x\|_1 = \sum \|x_i\|$ and $\|x\|_\infty = \max \|x_i\|$ for $x = (x_1, \ldots, x_n) \in X^n$.
(a) Construct isometric isomorphisms $\mathbb{K}_1^n \otimes_\pi X \cong X_1^n$ and $\mathbb{K}_\infty^n \otimes_\varepsilon X \cong X_\infty^n$.
(b) Identify $\mathbb{K}_1^n \otimes \mathbb{K}_\infty^n$ with the space $M_n(\mathbb{K})$ of $n \times n$-matrices via the isomorphism $x \otimes y \mapsto (x_i y_j)$. Given $a = (a_{ij}) \in M_n(\mathbb{K})$, calculate $\|a\|_\pi$ and $\|a\|_\varepsilon$ explicitly in terms of the matrix elements a_{ij}, and deduce that $\|\cdot\|_\pi \neq \|\cdot\|_\varepsilon$ unless $n = 1$.

9.13. Given a set I and a Banach space X, construct an isometric isomorphism $\ell^1(I) \mathbin{\widehat{\otimes}_\pi} X \cong \ell^1(I, X)$, where

$$\ell^1(I,X) = \Big\{ x = (x_i) \in X^I : \|x\| = \sum_i \|x_i\| < \infty \Big\}.$$

9.14. Let I, J be sets, and let P, Q be Köthe sets on I and J, respectively. Define a Köthe set $P \odot Q$ on $I \times J$ by letting $P \odot Q = \{(p_i q_j) : p \in P,\ q \in Q\}$. Construct a topological isomorphism $\lambda^1(I, P) \mathbin{\widehat{\otimes}_\pi} \lambda^1(J, Q) \cong \lambda^1(I \times J, P \odot Q)$.

9.15. Construct topological isomorphisms
(a) $s(\mathbb{Z}^n) \mathbin{\widehat{\otimes}_\pi} s(\mathbb{Z}^m) \cong s(\mathbb{Z}^{n+m})$ (see Exercise 8.6 for the definition of $s(\mathbb{Z}^n)$);
(b) $C^\infty(\mathbb{T}^m) \mathbin{\widehat{\otimes}_\pi} C^\infty(\mathbb{T}^n) \cong C^\infty(\mathbb{T}^{m+n})$, $\quad f \otimes g \mapsto \big((x,y) \mapsto f(x)g(y)\big)$.
(*Hint to* (b): use (a) and the Fourier transform.)

9.16. Construct a topological isomorphism $\mathscr{O}(\mathbb{D}^m_R) \mathbin{\widehat{\otimes}_\pi} \mathscr{O}(\mathbb{D}^n_S) \cong \mathscr{O}(\mathbb{D}^{m+n}_{(R,S)})$, $f \otimes g \mapsto \big((x,y) \mapsto f(x)g(y)\big)$. (*Hint*: use Exercises 9.14 and 8.7.)

9.17. (a) Given a set I and a Banach space X, construct an isometric isomorphism $c_0(I) \mathbin{\widehat{\otimes}_\varepsilon} X \cong c_0(I, X)$, where

$$c_0(I,X) = \Big\{ x = (x_i) \in X^I : \lim_{i\to\infty} \|x_i\| = 0 \Big\} \quad \text{with the norm } \|x\|_\infty = \sup_i \|x_i\|.$$

(b) Construct an isometric isomorphism $c_0(I) \mathbin{\widehat{\otimes}_\varepsilon} c_0(J) \cong c_0(I \times J)$, where $c_0(I) = c_0(I, \mathbb{K})$.

9.18. Prove that the canonical map $\ell^1 \mathbin{\widehat{\otimes}_\pi} c_0 \to \ell^1 \mathbin{\widehat{\otimes}_\varepsilon} c_0$ is neither topologically injective nor surjective.

9.19. Let X and Y be locally compact topological spaces. Construct a topological isomorphism $C(X) \mathbin{\widehat{\otimes}_\varepsilon} C(Y) \cong C(X \times Y)$, $\quad f \otimes g \mapsto \big((x,y) \mapsto f(x)g(y)\big)$. (*Hint*: see the lectures.)

9.20. Construct a topological isomorphism $C^\infty(\mathbb{R}^m) \mathbin{\widehat{\otimes}_\varepsilon} C^\infty(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^{m+n})$, $\quad f \otimes g \mapsto \big((x,y) \mapsto f(x)g(y)\big)$. (*Hint*: use the isomorphism $C(X) \mathbin{\widehat{\otimes}_\varepsilon} C(Y) \cong C(X \times Y)$ for compact spaces X, Y.)

9.21. Construct a topological isomorphism $\mathscr{S}(\mathbb{R}^m) \mathbin{\widehat{\otimes}_\varepsilon} \mathscr{S}(\mathbb{R}^n) \cong \mathscr{S}(\mathbb{R}^{m+n})$, $\quad f \otimes g \mapsto \big((x,y) \mapsto f(x)g(y)\big)$. (*Hint*: see the hint to the previous exercise.)

Given a Köthe set P on a set I, let $\lambda^0(I,P) = \Big\{ x = (x_i) \in \mathbb{K}^I : (x_i p_i) \in c_0(I)\ \forall p \in P \Big\}$.

9.22. (a) Prove that $\lambda^0(I, P)$ is a closed vector subspace of $\lambda^\infty(I, P)$. Hence $\lambda^0(I, P)$ as a complete locally convex space.
(b) Given Köthe sets P on I and Q on J, construct a topological isomorphism $\lambda^0(I, P) \mathbin{\widehat{\otimes}_\varepsilon} \lambda^0(J, Q) \cong \lambda^0(I \times J, P \odot Q)$ (cf. Exercise 9.14).

9.23. Prove directly (that is, without referring to the nuclearity of the spaces involved) that the canonical maps $C^\infty(\mathbb{T}^m) \mathbin{\widehat{\otimes}_\pi} C^\infty(\mathbb{T}^n) \to C^\infty(\mathbb{T}^m) \mathbin{\widehat{\otimes}_\varepsilon} C^\infty(\mathbb{T}^n)$ and $\mathscr{O}(\mathbb{D}^m_R) \mathbin{\widehat{\otimes}_\pi} \mathscr{O}(\mathbb{D}^n_S) \to \mathscr{O}(\mathbb{D}^m_R) \mathbin{\widehat{\otimes}_\varepsilon} \mathscr{O}(\mathbb{D}^n_S)$ are topological isomorphisms. (*Hint*: use Exercises 9.14 and 9.22 (b).)

Announcement. At Lecture 16, we will generalize the results of Exercises 9.15 (b), 9.16, 9.20, and 9.23 to the spaces of smooth (resp. holomorphic) functions on smooth real (resp. complex analytic) manifolds.