Köthe sequence spaces

(exercises for Lecture 11)

8.1. Prove that every Köthe space $\lambda^\nu(I,P)$ is complete.

8.2. Let P be a Köthe set on a set I, and let $\nu \in [1,+\infty]$.
(a) Assume that $p_i > 0$ for all $p \in P$ and all $i \in I$. Construct a topological isomorphism between $\lambda^\nu(I,P)$ and the projective limit of a family of the Banach spaces $\ell^\nu(I)$ where the connecting maps of the projective system are diagonal operators.
(b) Extend (a) to an arbitrary Köthe set P.

8.3. Consider the Köthe set $P = \{p^{(1)}, p^{(2)}, \ldots\}$ on $\mathbb{N}$, where $p^{(m)} = (1,\ldots,1,0,0,\ldots)$ (the first m entries are equal to 1, the other entries are 0).
(a) Show that for every $\nu \in [1,+\infty]$ we have $\lambda^\nu(\mathbb{N},P) = \mathbb{K}^\mathbb{N}$ topologically.
(b) Extend (a) to the space $\mathbb{K}^S$, where S is an arbitrary set.

8.4. Consider the Köthe set $P = \{p^{(1)}, p^{(2)}, \ldots\}$ on $\mathbb{N}$, where $p_k^{(m)} = k^m$. Show that for every $\nu \in [1,+\infty]$ we have $\lambda^\nu(\mathbb{N},P) = s$ topologically. (Recall that $\lambda^\infty(\mathbb{N},P) = s$ by definition, see Exercise sheet 2.)

8.5. Consider the Köthe set $P = \{p^{(1)}, p^{(2)}, \ldots\}$ on $\mathbb{Z}_{\geqslant 0}$, where $p_k^{(m)} = m^k$. Show that for every $\nu \in [1,+\infty]$ there is a topological isomorphism $\lambda^\nu(\mathbb{Z}_{\geqslant 0},P) \cong \mathscr{O}(\mathbb{C})$.

8.6. Consider the Köthe set $P = \{p^{(1)}, p^{(2)}, \ldots\}$ on $\mathbb{Z}^n$, where $p_k^{(m)} = (1+|k|)^m$. Show that for every $\nu \in [1,+\infty]$ we have $\lambda^\nu(\mathbb{Z}^n,P) = s(\mathbb{Z}^n)$ topologically. (We have $\lambda^\infty(\mathbb{Z}^n,P) = s(\mathbb{Z}^n)$ by definition, cf. Exercise sheet 2.)

8.7. Given $R = (R_1,\ldots,R_n) \in (0,+\infty]^n$, define a Köthe set P on $\mathbb{Z}^n_{\geqslant 0}$ by letting

$$P = \{p^{(r)} : r = (r_1,\ldots,r_n) \in (0,R_1) \times \cdots \times (0,R_n)\},$$

where $p_k^{(r)} = r_1^{k_1} \cdots r_n^{k_n}$ for $k = (k_1,\ldots,k_n) \in \mathbb{Z}^n_{\geqslant 0}$. Show that for every $\nu \in [1,+\infty]$ there is a topological isomorphism $\lambda^\nu(\mathbb{Z}^n_{\geqslant 0},P) \cong \mathscr{O}(\mathbb{D}^n_R)$, where $\mathbb{D}^n_R = \{z \in \mathbb{C}^n : |z_i| < R_i \ \forall i = 1,\ldots,n\}$ is the open polydisk in $\mathbb{C}^n$ of polyradius R.

8.8 (*the Aizenberg-Mityagin theorem*)**.** An open set D in $\mathbb{C}^n$ is a *complete Reinhardt domain* if for each $z = (z_1,\ldots,z_n) \in D$ and each $(\lambda_1,\ldots,\lambda_n) \in \mathbb{C}^n$ such that $|\lambda_i| \leqslant 1$ $(i = 1,\ldots,n)$ we have $(\lambda_1 z_1,\ldots,\lambda_n z_n) \in D$. Given a complete bounded Reinhardt domain D and $k = (k_1,\ldots,k_n) \in \mathbb{Z}^n_{\geqslant 0}$, let $b_k(D) = \sup_{z\in D} |z_1^{k_1} \cdots z_n^{k_n}|$. Define a Köthe set P on $\mathbb{Z}^n_{\geqslant 0}$ by letting

$$P = \{(b_k(D)s^{|k|})_{k\in\mathbb{Z}^n_{\geqslant 0}} : 0 < s < 1\}$$

where $|k| = k_1 + \cdots + k_n$ for $k = (k_1,\ldots,k_n) \in \mathbb{Z}^n_{\geqslant 0}$.
(a) Show that for every $\nu \in [1,+\infty]$ there is a topological isomorphism $\lambda^\nu(\mathbb{Z}^n_{\geqslant 0},P) \cong \mathscr{O}(D)$.
(b) Show that, if $D = \mathbb{D}^n_R$ is the open polydisk of polyradius R, then (a) yields the result of Exercise 8.7.
(c) Give an explicit form of (a) in the case where $D = \mathbb{B}^n_R = \{z \in \mathbb{C}^n : \sum |z_i|^2 < R^2\}$ is the open ball of radius R.

8.9. Given $z \in \mathbb{C}$, construct a topological isomorphism $\mathscr{O}_z \cong \lambda^\nu(\mathbb{Z}_{\geqslant 0},P)$, where P is a suitable Köthe set on $\mathbb{Z}_{\geqslant 0}$, and where $\nu \in [1,+\infty]$ is arbitrary. As a corollary, $\mathscr{O}_z$ is complete. (*Hint:* see Exercise 6.17 (a).)

8.10. Consider the vector space of *tempered sequences*

$$X = \Big\{ x = (x_n) \in \mathbb{K}^{\mathbb{N}} : \sup_{n \in \mathbb{N}} |x_n| n^{-k} < \infty \text{ for some } k \in \mathbb{N} \Big\}.$$

We equip X with the locally convex inductive limit topology of the sequence $(X_k)_{k \in \mathbb{N}}$ of Banach spaces, where

$$X_k = \Big\{ x = (x_n)_{n \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \|x\|_k = \sup_{n \in \mathbb{N}} |x_n| n^{-k} < \infty \Big\}$$

and the connecting maps $X_k \to X_\ell$ $(k \leqslant \ell)$ are the tautological inclusions.

(a) Let s' denote the strong dual of the space s of rapidly decreasing sequences. Show that the map

$$s' \to X, \quad F \mapsto (x_n(F) = F(e_n))_{n \in \mathbb{N}},$$

is a topological isomorphism (here e_n is the sequence with 1 in the nth slot, 0 elsewhere).

(b) Let P denote the family of all nonnegative sequences from s. Show that, for each $\nu \in [1, +\infty]$, we have $X = \lambda^\nu(\mathbb{N}, P)$ topologically.

8.11. Consider the vector space

$$X = \Big\{ x = (x_n) \in \mathbb{C}^{\mathbb{Z}_{\geqslant 0}} : \sup_{n \in \mathbb{N}} |x_n| r^{-n} < \infty \text{ for some } r > 0 \Big\}.$$

We equip X with the locally convex inductive limit topology of the family $(X_r)_{r>0}$ of Banach spaces, where

$$X_r = \Big\{ x = (x_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{Z}_{\geqslant 0}} : \|x\|_r = \sup_{n \in \mathbb{N}} |x_n| r^{-n} < \infty \Big\}$$

and the connecting maps $X_r \to X_s$ $(r \geqslant s)$ are the tautological inclusions.

(a) Let $\mathscr{O}(\mathbb{D})'$ denote the strong dual of the space $\mathscr{O}(\mathbb{D})$ of holomorphic functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Show that the map

$$\mathscr{O}(\mathbb{D})' \to X, \quad F \mapsto (x_n(F) = F(z^n/n!))_{n \in \mathbb{Z}_{\geqslant 0}},$$

is a topological isomorphism.

(b) Let P denote the family of all nonnegative sequences $p = (p_n)_{n \geqslant 0}$ such that the sequence $(p_n r^n)$ is bounded for all $r \in (0,1)$ (equivalently, such that the power series $\sum_n p_n z^n$ converges in $\mathbb{D}$). Show that, for each $\nu \in [1, +\infty]$, we have $X = \lambda^\nu(\mathbb{Z}_{\geqslant 0}, P)$ topologically.