# Completeness
(exercises for Lectures 8–9)

**7.1.** Let $X$ be a topological space, $Y \subset X$, and $x \in X$. Show that $x \in \overline{Y}$ iff there is a net in $Y$ which converges to $x$.

**7.2.** Let $X$ and $Y$ be topological spaces. Show that a map $f\colon X \to Y$ is continuous at $x \in X$ iff for every net $(x_\lambda)$ in $X$ such that $x_\lambda \to x$ we have $f(x_\lambda) \to f(x)$ in $Y$.

**7.3.** Show that a topological space $X$ is Hausdorff iff every net in $X$ has at most one limit.

**7.4.** Show that a topological space $X$ is compact iff every net in $X$ has an accumulation point.

**7.5.** Let $X$ be a topological vector space. Prove that
**(a)** each convergent net in $X$ is a Cauchy net;
**(b)** each Cauchy net in $X$ which has an accumulation point $x \in X$ converges to $x$.

**7.6.** Show that a continuous linear map between topological vector spaces takes Cauchy nets to Cauchy nets.

**7.7.** Let $(X, \tau(P))$ be a locally convex space. Show that a net $(x_\lambda)$ in $X$
**(a)** converges to $x \in X$ iff $p(x_\lambda - x) \to 0$ for every $p \in P$;
**(b)** is a Cauchy net iff for each $p \in P$ and each $\varepsilon > 0$ there exists $\lambda_0 \in \Lambda$ such that we have $p(x_\lambda - x_\mu) < \varepsilon$ whenever $\lambda, \mu \geqslant \lambda_0$.

**7.8.** Let $X$ be a vector space equipped with the projective locally convex topology generated by a family $\{\varphi_i\colon X \to X_i : i \in I\}$ of linear maps, where $\{X_i : i \in I\}$ is a family of locally convex spaces. Show that a net $(x_\lambda)$ in $X$ converges to $x \in X$ iff we have $\varphi_i(x_\lambda) \to \varphi_i(x)$ for all $i \in I$.

**7.9.** Show that a compact subset of a Hausdorff topological vector space is complete.

**7.10.** Let $X$ be a topological vector space, and let $X_0 \subset X$ be a dense vector subspace. Show that every continuous linear map from $X_0$ to a complete topological vector space $Y$ uniquely extends to a continuous linear map from $X$ to $Y$.

**7.11.** Let $X$ be a metrizable topological vector space, and let $\rho$ be a translation invariant metric on $X$ that generates the topology of $X$. Prove that the following conditions are equivalent:

(i) $X$ is complete;
(ii) $X$ is sequentially complete;
(iii) $(X, \rho)$ is a complete metric space.

**7.12.** Let $S$ be an uncountable set, and let $X$ be the subspace of $\mathbb{K}^S$ consisting of all countably supported functions. Prove that $X$ is sequentially complete, but is not complete.

**7.13.** Let $X$ be a topological vector space, and let $X_0 \subset X$ be a dense vector subspace. Show that
**(a)** every continuous seminorm $p$ on $X_0$ uniquely extends to a continuous seminorm $\tilde{p}$ on $X$;
**(b)** if $P$ is a defining family of seminorms on $X_0$, then $\{\tilde{p} : p \in P\}$ is a defining family of seminorms on $X$;
**(c)** if $\mathscr{U}$ is a base of neighborhoods of 0 in $X_0$, then $\{\overline{U} : U \in \mathscr{U}\}$ is a base of neighborhoods of 0 in $X$.

**7.14.** Let $(X_i)_{i \in I}$ be a family of locally convex spaces. Show that $\prod_{i \in I} X_i$ is complete iff all the spaces $X_i$ are complete.

**7.15.** Let $(X_i)_{i\in I}$ be a family of locally convex spaces. Show that $\bigoplus_{i\in I} X_i$ is complete iff all the spaces $X_i$ are complete. As a corollary, the strongest locally convex space is complete.

**7.16.** Show that the projective limit of a family of locally convex spaces is a closed subspace of their product. As a corollary, the projective limit of complete locally convex spaces is complete.

**7.17.** Let $X$ be a locally convex space, and let $Y$ be a vector subspace of $X$. Suppose that $Y$ is equipped with a locally convex topology that is stronger (=finer) than the topology induced from $X$. We say that $Y$ is *locally closed* in $X$ if there is a base of neighborhoods of 0 in $Y$ consisting of sets closed in $X$. Show that, if $X$ is complete and $Y$ is locally closed in $X$, then $Y$ is complete.

**7.18.** Prove that the following locally convex spaces are complete:
**(a)** $C(T)$, where $T$ is a locally compact topological space;
**(b)** $C^\infty(U)$, where $U \subset \mathbb{R}^n$ is an open set;
**(c)** the space $s$ of rapidly decreasing sequences;
**(d)** the Schwartz space $\mathscr{S}(\mathbb{R}^n)$;
**(e)** $\mathscr{O}(U)$, where $U \subset \mathbb{C}$ is an open set;
**(f)** $C_c(T)$, where $T$ is a second countable locally compact topological space;
**(g)** $C_c^\infty(U)$, where $U \subset \mathbb{R}^n$ is an open set.

**7.19.** Let $X$ be a complete locally convex space, and let $P$ be a directed defining family of seminorms on $X$. Recall (see the lecture) that there exists a topological isomorphism $X \cong \varprojlim \tilde{X}_p$, where $\tilde{X}_p$ (for every $p \in P$) is the completion of the normed space $X_p = (X/p^{-1}(0), \hat{p})$. Describe $\tilde{X}_p$ explicitly for **(0)** $X = \mathbb{K}^S$, where $S$ is a set, and for the spaces **(a)** – **(f)** of Exercise 7.18.

**7.20.** Let $X$ be a Hausdorff locally convex space. Describe explicitly the completion of the dual space $X'$ equipped with the weak$^*$ topology.

**7.21.** Given a locally convex space $X$, let $X^\infty$ (resp. $X_\infty$) denote the product (resp. the locally convex direct sum) of countably many copies of $X$. Let now $X$ and $Y$ be Banach spaces such that $Y$ is continuously embedded into $X$ and such that $Y$ is dense in $X$ (for example, $X = \ell^2$ and $Y = \ell^1$). Define

$$\varphi\colon X_\infty \oplus Y^\infty \to X^\infty, \quad (x, y) \mapsto x + y.$$

Prove that $\varphi$ is an open map onto a proper dense subspace of $X^\infty$. Deduce that the quotient $(X_\infty \oplus Y^\infty)/\operatorname{Ker}\varphi$ is incomplete (while $X_\infty \oplus Y^\infty$ itself is complete).

**7.22.** Let $T$ be a locally compact Hausdorff topological space, and let $S \subset T$ be a closed subset. Prove that

(i) the restriction map $r_S\colon C(T) \to C(S)$ is an open map onto a dense subspace of $C(S)$;
(ii) if $T$ is not normal, then there exists a closed set $S \subset T$ such that $r_S$ is not onto;
(iii) if $S$ is as in (ii), then the quotient $C(T)/\operatorname{Ker} r_S$ is incomplete (while $C(T)$ itself is complete).