Projective and inductive limits

(EXERCISES FOR LECTURES 6–7)

6.1. Let $\mathscr{X} = (X_i, \varphi_{ij})$ be a projective system of locally convex spaces indexed by a directed set I. Let

$$X = \Big\{ x = (x_i) \in \prod_{i \in I} X_i : x_i = \varphi_{ij}(x_j)\ \forall i < j \Big\}.$$

Equip X with the topology induced from $\prod_{i \in I} X_i$. For each $i \in I$, let $\varphi_i \colon X \to X_i$ denote the projection onto the ith factor.
(a) Show that (X, φ_i) is the (category-theoretic) projective limit of $\mathscr{X}$ in LCS.
(b) Show that X is closed in $\prod_{i \in I} X_i$.
(c) Suppose that β_i is a base of neighborhoods of 0 in X_i $(i \in I)$. Show that the family $\{\varphi_i^{-1}(U) : i \in I,\ U \in \beta_i\}$ is a base of neighborhoods of 0 in X (and is not just a subbase; compare with Exercise 5.1 (d)).

6.2. Let (X_i, φ_{ij}) be a projective system of locally convex spaces, $X = \varprojlim (X_i, \varphi_{ij})$, and let Y be a vector subspace of X. For each i, let $Y_i = \varphi_i(Y) \subset X_i$ (where $\varphi_i \colon X \to X_i$ is the canonical map).
(a) Show that $\overline{Y} = \varprojlim (\overline{Y_i}, \varphi_{ij}|_{\overline{Y_j}})$.
(b) Do we always have $Y = \varprojlim (Y_i, \varphi_{ij}|_{Y_j})$?

A projective system $\mathscr{X}$ of locally convex spaces is said to be *reduced* if all the canonical maps $\varprojlim \mathscr{X} \to X_i$ have dense ranges. A *reduced projective limit* is the projective limit of a reduced projective system.

6.3. Let $X = \varprojlim (X_i, \varphi_{ij})$ be a reduced projective limit of complete locally convex spaces. Prove that X is normable if and only if the system (X_i, φ_{ij}) "stabilizes at a normable space" in the following sense: there exists $i_0 \in I$ such that X_{i_0} is normable, and such that for all $j \geqslant i \geqslant i_0$ the connecting map $\varphi_{ij} \colon X_j \to X_i$ is a topological isomorphism.

6.4. Let $(X_i)_{i \in I}$ be a family of locally convex spaces. Construct a topological isomorphism $\prod_{i \in I} X_i \cong \varprojlim \{\prod_{i \in J} X_i : J \subset I \text{ is a finite subset}\}$.

6.5. Let X be a locally compact Hausdorff topological space.
(a) Construct a topological isomorphism $C(X) \cong \varprojlim \{C(K) : K \subset X \text{ is a compact set}\}$.
(b) Assume that X is second countable, and let $(K_j)_{j \in \mathbb{N}}$ be a compact exhaustion of X, i.e., a sequence of compact sets such that $X = \bigcup K_j$ and such that $K_j \subset \operatorname{Int} K_{j+1}$ for all j. (A subexercise: prove that a compact exhaustion exists.) Construct a topological isomorphism $C(X) \cong \varprojlim_{j \in \mathbb{N}} C(K_j)$.

6.6. Let $U \subset \mathbb{C}$ be an open set, and let $(K_j)_{j \in \mathbb{N}}$ be a compact exhaustion of U. For each j, let $\mathscr{A}(K_j)$ denote the subspace of $C(K_j)$ consisting of functions holomorphic on $\operatorname{Int} K_j$. Construct a topological isomorphism $\mathscr{O}(U) \cong \varprojlim_{j \in \mathbb{N}} \mathscr{A}(K_j)$.

6.7. Let $U \subset \mathbb{C}$ be an open set. Represent $\mathscr{O}(U)$ as the projective limit of a sequence of Hilbert spaces.

6.8. Define $\varphi \colon \ell^\infty \to \ell^\infty$ by $\varphi(x_1, x_2, \ldots) = (x_1, x_2/2, x_3/3, \ldots)$.
(a) Show that the projective limit of the sequence $\ell^\infty \xleftarrow{\varphi} \ell^\infty \xleftarrow{\varphi} \ell^\infty \xleftarrow{\varphi} \cdots$ is topologically isomorphic to s, the space of rapidly decreasing sequences (see Exercise sheet 2).
(b) Show that replacing ℓ^∞ by ℓ^p (where $1 \leqslant p < \infty$) or by c_0 yields the same projective limit.

6.9. Construct a topological isomorphism $C^\infty(\mathbb{R}) \cong \varprojlim_{k \in \mathbb{N}} C^k[-k, k]$.

6.10*. Represent $C^\infty(\mathbb{R})$ as the projective limit of a sequence of Hilbert spaces.

6.11. Represent the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ as the projective limit of a sequence of **(a)** Banach spaces; **(b)*** Hilbert spaces.

6.12. Let $\mathscr{X} = (X_i, \varphi_{ij})$ be an inductive system of locally convex spaces indexed by a directed set I. Let

$$X = \Big(\bigoplus_{i\in I} X_i\Big) / \operatorname{span}\{x - \varphi_{ij}(x) : i \leqslant j,\ x \in X_i\}.$$

For each $j \in I$, let $\varphi_j \colon X_j \to X$ denote the composite of the standard embedding $X_j \to \bigoplus_{i\in I} X_i$ and the quotient map $\bigoplus_{i\in I} X_i \to X$. Show that (X, φ_i) is the (category-theoretic) inductive limit of $\mathscr{X}$ in LCS.

6.13. **(a)** Let X be a vector space. Suppose that $X = \bigcup_{i\in I} X_i$, where $(X_i)_{i\in I}$ is a family of vector subspaces of X indexed by a directed set I such that $X_i \subset X_j$ whenever $i \leqslant j$. Suppose also that each X_i is equipped with a locally convex topology in such a way that the inclusions $X_i \hookrightarrow X_j$ are continuous for all $i \leqslant j$. Equip X with the inductive locally convex topology generated by the family $(X_i \hookrightarrow X)_{i\in I}$ of inclusions. Show that $X \cong \varinjlim X_i$.
(b) Prove that the inductive limit of every inductive system (X_i, φ_{ij}) of locally convex spaces such that all the φ_{ij}'s are injective can be obtained as in (a).

6.14. Let K be a compact subset of $\mathbb{C}^n$, and let $\mathscr{U}$ be a base of relatively compact open neighborhoods of K. Recall (see the lecture) that the space $\mathscr{O}(K)$ of germs of holomorphic functions on K is the locally convex inductive limit $\varinjlim\{\mathscr{O}(U) : U \in \mathscr{U}\}$. Construct a topological isomorphism $\mathscr{O}(K) \cong \varinjlim\{\mathscr{A}(\overline{U}) : U \in \mathscr{U}\}$, where $\mathscr{A}(\overline{U})$ is defined in Exercise 6.6.

6.15. Let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$. Show that the restriction map $\mathscr{O}(\mathbb{D}_R) \to \mathscr{O}(\mathbb{D}_r)$ (where $R > r$) is not topologically injective. As a corollary, the inductive sequence $\{\mathscr{O}(\mathbb{D}_{1/n}) : n \in \mathbb{N}\}$ is not strict.

6.16. Prove that the following spaces are Hausdorff: **(a)** $C_c(X)$ (where X is a topological space); **(b)** $C_c^\infty(U)$ (where U is an open subset of $\mathbb{R}^n$); **(c)** $\mathscr{O}(K)$ (where K is a compact subset of $\mathbb{C}^n$). (In (a) and (b), try to avoid using general results on strict inductive limits. In (c), the respective inductive limit is not strict by Exercise 6.15.)

6.17. Given $z \in \mathbb{C}$, let $\mathscr{O}_z$ denote the space of germs of holomorphic functions at z (i.e., $\mathscr{O}_z = \mathscr{O}(\{z\})$, see Exercise 6.14).
(a) Let P denote the set of all sequences $p = (p_n)$, $p_n \geqslant 0$, such that $p_n = o(\varepsilon^n)$ for every $\varepsilon > 0$. For each $p \in P$, define a seminorm $\|\cdot\|_p$ on $\mathscr{O}_z$ by $\|f\|_p = \sum_n |c_n(f)| p_n$, where $c_n(f) = f^{(n)}(z)/n!$. Show that the family $\{\|\cdot\|_p : p \in P\}$ of seminorms is defining for $\mathscr{O}_z$.
(b) Show that a sequence (f_n) converges in $\mathscr{O}_z$ iff there is a neighborhood $U \ni z$ such that (f_n) is contained in $\mathscr{O}(U)$ and converges uniformly on U.
(c) Is $\mathscr{O}_z$ metrizable?

6.18. Let $\mathscr{C}_0 = \varinjlim C[-1/n, 1/n]$ be the space of germs of continuous functions at $0 \in \mathbb{R}$ equipped with the respective inductive locally convex topology, and let $E = \{f \in \mathscr{C}_0 : f(0) = 0\}$. Prove that the topology on E induced from $\mathscr{C}_0$ is anti-discrete. As a corollary, $\mathscr{C}_0$ is not Hausdorff.

6.19. Let $\mathscr{O}_0$ denote the space of germs of holomorphic functions at $0 \in \mathbb{C}$. Write $\mathscr{O}_0$ in the form $\mathscr{O}_z = \varinjlim \mathscr{O}(\mathbb{D}_{1/n})$, where $\mathbb{D}_{1/n} = \{z \in \mathbb{C} : |z| < 1/n\}$. Let us now change the canonical topology on $\mathscr{O}_0$ as follows: equip each $\mathscr{O}(\mathbb{D}_{1/n})$ with the topology of pointwise convergence, and equip $\mathscr{O}_0$ with the respective inductive locally convex topology. Let $E = \{f \in \mathscr{O}_0 : f(0) = 0\}$. Prove that the topology on E induced from $\mathscr{O}_0$ is anti-discrete. As a corollary, $\mathscr{O}_0$ is not Hausdorff for the above (nonstandard) topology.

6.20. Let X be the strict inductive limit of a sequence (X_n) of locally convex spaces. Suppose that each X_n is closed in X_{n+1}. Prove that $B \subset X$ is bounded iff there exists n such that $B \subset X_n$ and such that B is bounded in X_n.