Projective and inductive locally convex topologies. Products and coproducts

(exercises for Lectures 4–5)

5.1. Let X be a vector space equipped with the projective locally convex topology generated by a family of linear maps $(\varphi_i\colon X \to X_i)_{i\in I}$, where $(X_i)_{i\in I}$ is a family of locally convex spaces. Show that
(a) the projective topology on X is the weakest locally convex topology on X that makes all the maps φ_i continuous;
(b) the projective topology on X is the weakest topology on X that makes all the maps φ_i continuous;
(c) the projective topology on X is a unique locally convex topology on X having the following property: if Y is a locally convex space, then a linear map $\psi\colon Y \to X$ is continuous if and only if all the maps $\varphi_i \circ \psi\colon Y \to X_i$ are continuous;
(d) if σ_i is a neighborhood subbase at 0 in X_i ($i \in I$), then the family $\{\varphi_i^{-1}(U) : i \in I,\ U \in \sigma_i\}$ is a neighborhood subbase at 0 in X.

5.2. Let X be a vector space equipped with the inductive locally convex topology generated by a family of linear maps $(\varphi_i\colon X_i \to X)_{i\in I}$, where $(X_i)_{i\in I}$ is a family of locally convex spaces. Show that
(a) the inductive topology on X is the strongest locally convex topology on X that makes all the maps φ_i continuous;
(b) the inductive topology on X is a unique locally convex topology on X having the following property: if Y is a locally convex space, then a linear map $\psi\colon X \to Y$ is continuous if and only if all the maps $\psi \circ \varphi_i\colon X_i \to Y$ are continuous;
(c) the family of all absorbing, absolutely convex sets $U \subset X$ such that, for each $i \in I$, $\varphi^{-1}(U_i)$ is a neighborhood of zero in X_i, is a neighborhood base at zero in X.

5.3. Let X, X_i, φ_i be as in Exercise 5.2. Assume also that $X = \sum_{i\in I} \varphi_i(X_i)$. Let $\tau_{\text{ind.lcs}}$ denote the respective inductive locally convex topology on X.
(a) Prove that there exists the strongest topology on X that makes all the maps φ_i continuous, and describe it explicitly. This topology will be denoted by $\tau_{\text{ind.top}}$.
(b) Prove that there exists the strongest vector space topology on X that makes all the maps φ_i continuous, and describe it explicitly. This topology will be denoted by $\tau_{\text{ind.tvs}}$.
(c)* Prove that, if I is at most countable, then $\tau_{\text{ind.lcs}} = \tau_{\text{ind.tvs}}$. (*Hint*: it suffices to show that sets of the form $\sum_i \varphi_i(U_i)$, where $U_i \subset X_i$ is a neighborhood of 0, form a neighborhood base at 0 in $\tau_{\text{ind.tvs}}$.)
(d)* Construct an example such that I is uncountable and $\tau_{\text{ind.lcs}} \neq \tau_{\text{ind.tvs}}$. (*Hint*: consider the strongest locally convex topology on a vector space of uncountable dimension.)
(e) Construct an example such that I is finite and $\tau_{\text{ind.tvs}} \neq \tau_{\text{ind.top}}$.

5.4. (a) Show that the product of a family of locally convex spaces is their product in LCS (in the category-theoretic sense).
(b) Show that an infinite family of nonzero normed spaces does not have a product in the category of normed spaces and continuous linear maps.

5.5. Let $(X_i)_{i\in I}$ be a family of nonzero locally convex spaces. Show that
(a) $\prod_{i\in I} X_i$ is Hausdorff $\iff$ all the X_i's are Hausdorff;
(b) $\prod_{i\in I} X_i$ is normable $\iff$ all the X_i's are normable, and I is finite;
(c) $\prod_{i\in I} X_i$ is metrizable $\iff$ all the X_i's are metrizable, and I is at most countable.

5.6. Let X be a vector space equipped with the projective topology generated by a family $(\varphi_i\colon X \to X_i)_{i\in I}$ of linear maps, where $(X_i)_{i\in I}$ is a family of locally convex spaces. Prove that a set $B \subset X$ is bounded if and only if $\varphi_i(B)$ is bounded in X_i for all $i \in I$. In particular, a set $B \subset \prod_{i\in I} X_i$ is bounded if and only if $B \subset \prod_{i\in I} B_i$, where $B_i \subset X_i$ are bounded sets.

5.7. (a) Show that the locally convex direct sum of a family of locally convex spaces is their coproduct in LCS (in the category-theoretic sense).
(b) Show that an infinite family of nonzero normed spaces does not have a coproduct in the category of normed spaces and continuous linear maps.

5.8. Let $(X_i)_{i\in I}$ be a family of locally convex spaces. For each $i \in I$ choose a directed defining family P_i of seminorms on X_i. For each $p = (p_i) \in \prod_{i\in I} P_i$ and each $a = (a_i) \in [0, +\infty)^I$, define a seminorm $q_{a,p}$ on $\bigoplus_{i\in I} X_i$ by $q_{a,p}(x) = \sum_i a_i p_i(x_i)$ (where $x = \sum_i x_i$, $x_i \in X_i$). Show that
(a) $\left\{q_{a,p} : p \in \prod_{i\in I} P_i,\ a \in [0, +\infty)^I\right\}$ is a defining family of seminorms on $\bigoplus_{i\in I} X_i$;
(b) if each P_i is stable under multiplication by positive numbers (for example, if P_i consists of all continuous seminorms on X_i), then it suffices to consider seminorms of the form $q_{1,p}$ (where $p \in \prod_{i\in I} P_i$).

5.9. Let $(X_i)_{i\in I}$ be a family of locally convex spaces. Prove that
(a) if I is finite, then $\bigoplus_{i\in I} X_i = \prod_{i\in I} X_i$ as topological vector spaces;
(b) if I is infinite, and if the topology on X_i is nontrivial for all $i \in I$, then the standard (inductive) topology on $\bigoplus_{i\in I} X_i$ is strictly stronger than the topology induced from $\prod_{i\in I} X_i$.

5.10. Let $(X_i)_{i\in I}$ be a family of nonzero locally convex spaces. Show that
(a) $\bigoplus_{i\in I} X_i$ is Hausdorff $\iff$ all the X_i's are Hausdorff;
(b) $\bigoplus_{i\in I} X_i$ is normable $\iff$ all the X_i's are normable, and I is finite;
(c) $\bigoplus_{i\in I} X_i$ is metrizable $\iff$ all the X_i's are metrizable, and I is finite.
As a corollary (see Exercise 3.8), an infinite-dimensional strongest locally convex space is not metrizable.

5.11. Let $(X_i)_{i\in I}$ be a family of Hausdorff locally convex spaces. Show that a set $B \subset \bigoplus_{i\in I} X_i$ is bounded if and only if there exists a finite subset $J \subset I$ such that $B \subset \prod_{j\in J} B_j$, where $B_j \subset X_j$ are bounded sets.

5.12. Let X be a locally compact, second countable Hausdorff topological space, and let $C_c(X)$ be the space of compactly supported continuous functions on X topologized in the standard way. Let $C(X)_{\geqslant 0}$ denote the set of all nonnegative continuous functions on X. Given $a \in C(X)_{\geqslant 0}$, define a seminorm $\|\cdot\|_a$ on $C_c(X)$ by letting $\|f\|_a = \sup_{x\in X} |f(x)|a(x)$. Show that the family $\{\|\cdot\|_a : a \in C(X)_{\geqslant 0}\}$ of seminorms is defining for $C_c(X)$.

5.13. Let $U \subset \mathbb{R}^n$ be an open set, and let $C_c^\infty(U)$ be the space of compactly supported smooth functions on U topologized in the standard way. Let $\mathscr{V}$ denote the set of all tuples of the form $v = (v_\alpha)_{\alpha\in\mathbb{Z}^n_{\geqslant 0}}$, where $v_\alpha \in C(U)_{\geqslant 0}$ and the family $(\operatorname{supp} v_\alpha)_{\alpha\in\mathbb{Z}^n_{\geqslant 0}}$ is locally finite[1]. For each $v = (v_\alpha) \in \mathscr{V}$ define a seminorm $\|\cdot\|_v$ on $C_c^\infty(U)$ by letting

$$\|f\|_v = \sup_{\alpha\in\mathbb{Z}^n_{\geqslant 0}} \sup_{x\in U} |D^\alpha f(x)| v_\alpha(x).$$

Show that the family $\{\|\cdot\|_v : v \in \mathscr{V}\}$ of seminorms is defining for $C_c^\infty(U)$.

[1] A family $(X_i)_{i\in I}$ of subsets of a topological space X is *locally finite* if each $x \in X$ has a neighborhood U such that $U \cap X_i = \varnothing$ for all but finitely many $i \in I$.