Quotients, kernels, cokernels

(EXERCISES FOR LECTURE 4)

Notation. Let LCS denote the category of locally convex spaces and continuous linear maps, and let HLCS denote the full subcategory of LCS consisting of Hausdorff spaces.

4.1. Let X be a topological vector space, let $X_0 \subset X$ be a vector subspace, and let $Q\colon X \to X/X_0$ denote the quotient map. Show that

(a) Q is continuous and open;

(b) if β is a neighborhood base at 0 in X, then $\{Q(U) : U \in \beta\}$ is a base at 0 in X/X_0;

(c) the quotient X/X_0 is Hausdorff if and only if X_0 is closed in X.

4.2. Let X be a locally convex space, P be a directed defining family of seminorms on X, and X_0 be a vector subspace of X. Show that the family $\hat{P} = \{\hat{p} : p \in P\}$ of quotient seminorms is a defining family on X/X_0.

4.3. Let p be a seminorm on a vector space X, let X_0 be a vector subspace of X such that $X_0 \subset p^{-1}(0)$, and let $\hat{p}$ denote the quotient seminorm on X/X_0. Show that $\hat{p}(x + X_0) = p(x)$ for all $x \in X$.

4.4. Show that the inclusion functor $\mathsf{HLCS} \hookrightarrow \mathsf{LCS}$ has a left adjoint, and describe it explicitly. (*Hint:* consider $X_h = X/\overline{\{0\}}$.)

4.5. **(a)** Show that the kernel of a morphism $\varphi\colon X \to Y$ in LCS is the subspace $\varphi^{-1}(0)$, and that the cokernel of φ is the quotient $Y/\varphi(X)$.

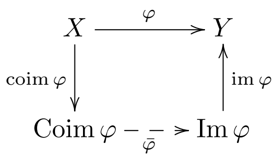
(b) Describe kernels and cokernels of morphisms in HLCS.

Let $\mathscr{A}$ be a category having a zero object. A morphism $\varphi\colon X \to Y$ in $\mathscr{A}$ is a *kernel* (resp., a *cokernel*) if there exists a morphism $\psi\colon Y \to Z$ (resp., $\psi\colon Z \to X$) such that $\varphi = \ker \psi$ (resp., $\varphi = \operatorname{coker} \psi$).

4.6. **(a)** Show that a morphism φ in LCS is a kernel if and only if it is topologically injective, and that φ is a cokernel if and only if it is open.

(b) Obtain a similar characterization of kernels and cokernels in HLCS.

Let $\mathscr{A}$ be a category having a zero object. Suppose that each morphism in $\mathscr{A}$ has a kernel and a cokernel. We define the *image* $(\operatorname{Im} \varphi, \operatorname{im} \varphi)$ of a morphism φ in $\mathscr{A}$ to be the kernel of the cokernel of φ, and the *coimage* $(\operatorname{Coim} \varphi, \operatorname{coim} \varphi)$ of φ to be the cokernel of the kernel of φ. Thus for each $\varphi\colon X \to Y$ there is a unique $\bar{\varphi}\colon \operatorname{Coim} \varphi \to \operatorname{Im} \varphi$ making the following diagram commute:



We say that φ is *strict* if $\bar{\varphi}$ is an isomorphism.

4.7. **(a)** Describe the image and the coimage of each morphism in the categories LCS and HLCS.

(b) Show that a morphism $\varphi\colon X \to Y$ in LCS is strict if and only if φ is an open map of X onto $\varphi(X)$.

(c) Describe strict morphisms in HLCS.