Topological vector spaces. Seminorms and convexity

(exercises for Lectures 1–2)

Convention. All vector spaces are over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

1.1. Let X be a topological vector space. Show that the closure of a vector subspace $X_0 \subset X$ is a vector subspace as well.

1.2. Let X and Y be topological vector spaces. Show that
(1) a linear map $X \to Y$ is continuous iff it is continuous at 0;
(2) the set $\mathscr{L}(X, Y)$ of all continuous linear maps from X to Y is a vector subspace of $\mathrm{Hom}_{\mathbb{K}}(X, Y)$.

1.3. Let (X, P) be a polynormed space. Show that the topology on X generated by P makes X into a topological vector space.
Hint: the shortest way is to reduce everything to seminormed spaces.

1.4. Let (X, P) be a polynormed space. Show that a sequence (x_n) in X converges to $x \in X$ w.r.t. the topology generated by P iff for all $p \in P$ we have $p(x_n - x) \to 0$.

1.5. Let (X, P) be a polynormed space. Show that $\overline{\{0\}} = \bigcap\{p^{-1}(0) : p \in P\}$.

1.6. Give a reasonable definition of the canonical topology on $C^\infty(M)$, where M is a smooth manifold. (This was done at the lecture in the special cases where M is either a closed interval on $\mathbb{R}$ or an open subset of $\mathbb{R}^n$.)

1.7. Let $U \subset \mathbb{C}$ be an open set. Show that the topology of compact convergence on the space $\mathscr{O}(U)$ of holomorphic functions is the same as the topology inherited from $C^\infty(U)$.

1.8. Let X be a vector space.
(a) Show that $S \subset X$ is convex iff for all $\lambda, \mu \geqslant 0$ we have $(\lambda + \mu)S = \lambda S + \mu S$.
(b) Give a similar characterization of absolutely convex sets.

Given a subset S of a vector space X, the *convex hull* of S is defined to be the intersection of all convex sets containing S. The convex hull of S is denoted by $\mathrm{conv}(S)$. The *circled hull*, $\mathrm{circ}(S)$, an the *absolutely convex hull*, $\Gamma(S)$, are defined similarly.

1.9. Let X be a vector space, and let $S \subset X$. Show that
(1) $\mathrm{conv}(S) = \Big\{\sum_{i=1}^n \lambda_i x_i : x_i \in S,\ \lambda_i \geqslant 0,\ \sum_{i=1}^n \lambda_i = 1,\ n \in \mathbb{N}\Big\}$;
(2) $\mathrm{circ}(S) = \Big\{\lambda x : x \in S,\ \lambda \in \mathbb{K},\ |\lambda| \leqslant 1\Big\}$;
(3) $\Gamma(S) = \Big\{\sum_{i=1}^n \lambda_i x_i : x_i \in S,\ \lambda_i \in \mathbb{K},\ \sum_{i=1}^n |\lambda_i| \leqslant 1,\ n \in \mathbb{N}\Big\}$.

1.10. Let X be a topological vector space, and let $S \subset X$. Show that
(1) if S is convex, then the closure $\overline{S}$ and the interior $\mathrm{Int}\, S$ are convex;
(2) if S is circled, then $\overline{S}$ is circled; if, moreover, $0 \in \mathrm{Int}\, S$, then $\mathrm{Int}\, S$ is circled;
(3) if S is open, then $\mathrm{conv}(S)$ and $\Gamma(S)$ are open; if, moreover, $0 \in S$, then $\mathrm{circ}(S)$ is open.

1.11. Let p and q be seminorms on a vector space. Show that $p \leqslant q$ iff $U_q \subset U_p$, and that $p \prec q$ iff $U_q \prec U_p$.

1.12. Show that any seminorm p on a vector space X equals the Minkowski functional of the open ball $U_p = \{x \in X : p(x) < 1\}$ and of the closed ball $\overline{U}_p = \{x \in X : p(x) \leqslant 1\}$.

1.13. Let S and T be absolutely convex, absorbing subsets of a vector space, and let p_S, p_T denote their Minkowski functionals. Prove that $p_S = p_T \iff U_{p_S} \subset T \subset \overline{U}_{p_S} \iff U_{p_T} \subset S \subset \overline{U}_{p_T}$.

1.14. Let X be a topological vector space, let $V \subset X$ be an absolutely convex neighborhood of 0, and let p_V denote the Minkowski functional of V. Show that $\operatorname{Int} V = \{x : p_V(x) < 1\}$ and $\overline{V} = \{x : p_V(x) \leqslant 1\}$. Deduce that $V \mapsto p_V$ is a 1-1-correspondence between the collection of all absolutely convex open neighborhoods of 0 and the collection of all continuous seminorms on X. Moreover, the inverse map is given by $p \mapsto U_p$.

1.15. Let X be a vector space. A function $p\colon X \to [0, +\infty)$ is an *F-seminorm*[1] if

1) $p(x + y) \leqslant p(x) + p(y) \quad (x, y \in X)$;
2) $p(\lambda x) \leqslant p(x) \quad (x \in X,\ |\lambda| \leqslant 1)$;
3) if (λ_n) is a sequence in $\mathbb{K}$ and $\lambda_n \to 0$, then for every $x \in X$ we have $p(\lambda_n x) \to 0$.

If, moreover, $p(x) > 0$ whenever $x \neq 0$, then p is an *F-norm*. Prove that, for every F-seminorm p on X, the topology on X generated by the semimetric $\rho(x, y) = p(x - y)$ makes X into a topological vector space.

1.16. Let (X, μ) be a σ-finite measure space, and let $0 < p < 1$. We define $L^p(X, \mu)$ to be the space of (μ-equivalence classes of) measurable functions $f\colon X \to \mathbb{K}$ such that $|f|^p$ is integrable. Given $f \in L^p(X, \mu)$, let

$$|f|_p = \int_X |f(x)|^p \, d\mu(x).$$

(a) Show that $|\cdot|_p$ is an F-norm on $L^p(X, \mu)$. Thus $L^p(X, \mu)$ is a metrizable topological vector space.
(b) Show that the only continuous linear functional on $L^p[0, 1]$ is identically zero. As a corollary, $L^p[0, 1]$ is not locally convex.
(c) Can $L^p(X, \mu)$ be locally convex and infinite-dimensional?

1.17. Let (X, μ) be a finite measure space. We define $L^0(X, \mu)$ to be the space of (μ-equivalence classes of) all measurable functions $f\colon X \to \mathbb{K}$. Choose a bounded monotone function $\varphi\colon [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

1) $\varphi(s + t) \leqslant \varphi(s) + \varphi(t) \quad (s, t \geqslant 0)$;
2) $\varphi(0) = 0$;
3) φ is a homeomorphism between suitable neighborhoods of 0.

For example, we can let $\varphi(t) = t/(1 + t)$ or $\varphi(t) = \min\{t, 1\}$. Given $f \in L^0(X, \mu)$, let

$$|f|_0 = \int_X \varphi(|f(x)|) \, d\mu(x).$$

(a) Show that $|\cdot|_0$ is an F-norm on $L^0(X, \mu)$. Thus $L^0(X, \mu)$ is a metrizable topological vector space.
(b) Show that a sequence in $L^0(X, \mu)$ converges iff it converges in measure.
(c) Show that the only continuous linear functional on $L^0[0, 1]$ is identically zero. As a corollary, $L^0[0, 1]$ is not locally convex.
(d) Can $L^0(X, \mu)$ be locally convex and infinite-dimensional?

[1] "F" is for "Fréchet".