5.1. Let X be a topological vector space, let $X_0 \subset X$ be a vector subspace, and let $\pi\colon X \to X/X_0$ denote the quotient map. Show that
(a) if $\mathscr{U}$ is a neighborhood base at 0 in X, then $\{\pi(U) : U \in \mathscr{U}\}$ is a neighborhood base at 0 in X/X_0;
(b) the quotient X/X_0 is Hausdorff if and only if X_0 is closed in X.

5.2. Let X be a locally convex space, P be a directed fundamental family of seminorms on X, and X_0 be a vector subspace of X. Show that the family $\hat{P} = \{\hat{p} : p \in P\}$ of quotient seminorms is a fundamental family on X/X_0.

5.3. Let X be a topological vector space, let p be a continuous seminorm on X, and let $\hat{p}$ denote the quotient seminorm on $X/\overline{\{0\}}$. Show that $\hat{p}(x + \overline{\{0\}}) = p(x)$ $(x \in X)$.

5.4. (a) Show that the kernel of a morphism $\varphi\colon X \to Y$ in the category LCS of all locally convex spaces is the subspace $\varphi^{-1}(0)$, and that the cokernel of φ is the quotient $Y/\varphi(X)$.
(b) Describe kernels and cokernels of morphisms in the category HLCS of all Hausdorff locally convex spaces.

5.5. (a) Show that a morphism φ in LCS is a kernel if and only if it is topologically injective, and that φ is a cokernel if and only if it is open.
(b) Obtain a similar characterization of kernels and cokernels in HLCS.

Let $\mathscr{A}$ be a category having a zero object. Suppose that each morphism in $\mathscr{A}$ has a kernel and a cokernel. We define the *image* $(\operatorname{Im}\varphi, \operatorname{im}\varphi)$ of a morphism φ in $\mathscr{A}$ to be the kernel of the cokernel of φ, and the *coimage* $(\operatorname{Coim}\varphi, \operatorname{coim}\varphi)$ of φ to be the cokernel of the kernel of φ. Thus for each $\varphi\colon X \to Y$ there is a unique $\bar{\varphi}\colon \operatorname{Coim}\varphi \to \operatorname{Im}\varphi$ making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & Y \\ {\scriptstyle \operatorname{coim}\varphi}\downarrow & & \uparrow{\scriptstyle \operatorname{im}\varphi} \\ \operatorname{Coim}\varphi & \xrightarrow[\bar{\varphi}]{} & \operatorname{Im}\varphi \end{array}$$

We say that φ is *strict* if $\bar{\varphi}$ is an isomorphism.

5.6. (a) Describe the image and the coimage of each morphism in the categories LCS and HLCS.
(b) Show that a morphism $\varphi\colon X \to Y$ in LCS is strict in the above sense if and only if it is strict as a continuous linear map (see the lectures), i.e., if and only if φ is an open map of X onto $\varphi(X)$.
(c) Describe strict morphisms in HLCS.

5.7. Let X be a vector space equipped with the projective locally convex topology generated by a family of linear maps $(\varphi_i\colon X \to X_i)_{i\in I}$, where $(X_i)_{i\in I}$ is a family of locally convex spaces. Show that
(a) the projective topology on X is the weakest locally convex topology on X that makes all the maps φ_i continuous;
(b) the projective topology on X is the weakest topology on X that makes all the maps φ_i continuous;
(c) the projective topology on X is a unique locally convex topology on X having the following property: if Y is a locally convex space, then a linear map $\psi\colon Y \to X$ is continuous if and only if all the maps $\varphi_i \circ \psi\colon Y \to X_i$ are continuous;
(d) if σ_i is a neighborhood subbase at 0 in X_i $(i \in I)$, then the family $\{\varphi_i^{-1}(U_i) : U_i \in \sigma_i,\ i \in I\}$ is a neighborhood subbase at 0 in X.