4.1. Show that a relatively compact subset of a topological vector space is bounded.

4.2*. Prove that a Hausdorff topological vector space is locally compact if and only if it is finite-dimensional. (This result was proved at the lectures in the special case of locally convex spaces.)

4.3. Construct a linear map T between locally convex spaces X and Y which takes bounded subsets of X to bounded subsets of Y, but which is not sequentially continuous. Using this, find an example of a nonbornological locally convex space.

4.4*. Construct a discontinuous, sequentially continuous linear map between locally convex spaces.

4.5. Let LCS denote the category of locally convex spaces, and let CBorn denote the category of convex bornological spaces. Show that the functors $\mathrm{vN}\colon \mathsf{LCS} \to \mathsf{CBorn}$ and $\mathrm{top}\colon \mathsf{CBorn} \to \mathsf{LCS}$ are indeed functors (see the lecture), and construct a natural isomorphism

$$\mathrm{Hom}_{\mathsf{LCS}}(\mathrm{top}(X), Y) \cong \mathrm{Hom}_{\mathsf{CBorn}}(X, \mathrm{vN}(Y)) \qquad (X \in \mathsf{CBorn},\ Y \in \mathsf{LCS})$$

(in other words, $(\mathrm{top}, \mathrm{vN})$ is an adjoint pair of fucntors).

4.6. Let X and Y be topological vector spaces, and let $\mathscr{U}$ be a neighborhood subbase at 0 in X. Show that a linear map $\varphi\colon X \to Y$ is open if and only if for each $U \in \mathscr{U}$ $\varphi(U)$ is a neighborhood of 0 in Y.

4.7. Let X and Y be locally convex spaces, and let P and Q be fundamental families of seminorms on X and Y, respectively. Let $\varphi\colon X \to Y$ be a continuous linear map. Show that
(a) φ is topologically injective if and only if it is injective, and for each $p \in P$ there exist $c > 0$ and $q_1, \ldots, q_n \in Q$ such that $\max_{1 \leqslant i \leqslant n} q_i(\varphi(x)) \geqslant cp(x)$ $(x \in X)$. Moreover, if X is Hausdorff, then the latter condition implies the injectivity of φ.
(b) φ is open if and only if for each $p \in P$ there exist $C > 0$ and $q_1, \ldots, q_n \in Q$ such that for each $y \in Y$ there exists $x \in X$ satisfying $\varphi(x) = y$ and $p(x) \leqslant C \max_{1 \leqslant i \leqslant n} q_i(y)$.