3.1. Let s be the space of rapidly decreasing sequences (see Exercises for Lecture 2). Show that the following families of seminorms on s are equivalent:

(1) $\|x\|_k^{(\infty)} = \sup_n |x_n| n^k \quad (k \in \mathbb{Z}_{\geqslant 0})$;

(2) $\|x\|_k^{(1)} = \sum_n |x_n| n^k \quad (k \in \mathbb{Z}_{\geqslant 0})$;

(3) $\|x\|_k^{(p)} = \Big(\sum_n |x_n|^p n^{kp}\Big)^{1/p} \quad (k \in \mathbb{Z}_{\geqslant 0})$.

3.2. Sjow that the following families of seminorms on the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ (see Exercises for Lecture 2) are equivalent:

(1) $\|f\|_{\alpha,\beta} = \sup\limits_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)| \quad (\alpha,\beta \in \mathbb{Z}^n_{\geqslant 0})$;

(2) $\|f\|_{k,\beta} = \sup\limits_{x\in\mathbb{R}^n} \|x\|^k |D^\beta f(x)| \quad (k \in \mathbb{Z}_{\geqslant 0},\ ,\beta \in \mathbb{Z}^n_{\geqslant 0})$;

(3) $\|f\|^{(0)}_{k,\beta} = \sup\limits_{x\in\mathbb{R}^n} (1+\|x\|)^k |D^\beta f(x)| \quad (k \in \mathbb{Z}_{\geqslant 0},\ ,\beta \in \mathbb{Z}^n_{\geqslant 0})$;

(4) $\|f\|^{(1)}_{k,\beta} = \int_{\mathbb{R}^n} (1+\|x\|)^k |D^\beta f(x)|\, dx \quad (k \in \mathbb{Z}_{\geqslant 0},\ ,\beta \in \mathbb{Z}^n_{\geqslant 0})$;

(5) $\|f\|^{(p)}_{k,\beta} = \Big(\int_{\mathbb{R}^n} (1+\|x\|)^{kp} |D^\beta f(x)|^p\, dx\Big)^{1/p} \quad (k \in \mathbb{Z}_{\geqslant 0},\ ,\beta \in \mathbb{Z}^n_{\geqslant 0})$.

3.3. Let U be a domain in $\mathbb{C}$, and let $\mathscr{O}(U)$ denote the space of holomorphic functions on U. Choose a compact exhaustion $\{U_i\}_{i\in\mathbb{N}}$ of U (i.e., $U = \bigcup_i U_i$, U_i is open, $\overline{U_i}$ is compact, and $\overline{U_i} \subset U_{i+1}$ for all i). Let $p \in [1,+\infty)$, and let μ denote the Lebesgue measure on $\mathbb{C}$. Show that the following families of seminorms on $\mathscr{O}(U)$ are equivalent:

(1) $\|f\|_K = \sup\limits_{z\in K} |f(z)| \quad (K \subset U \text{ is a compact set})$;

(2) $\|f\|_{k,\ell,K} = \sup\limits_{z=x+iy\in K} \dfrac{\partial^{k+\ell} f(z)}{\partial x^k\, \partial y^\ell} \quad (K \subset U \text{ is a compact set, } k,\ell \in \mathbb{Z}_{\geqslant 0})$;

(3) $\|f\|_i^{(1)} = \int_{U_i} |f(z)|\, d\mu(z) \quad (i \in \mathbb{N})$;

(4) $\|f\|_i^{(p)} = \Big(\int_{U_i} |f(z)|^p\, d\mu(z)\Big)^{1/p} \quad (i \in \mathbb{N})$.

The equivalence of (1) and (2) means that the topology of compact convergence and the topology inherited from $C^\infty(U)$ are the same on $\mathscr{O}(U)$.

3.4. Let $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$. Given $f \in \mathscr{O}(\mathbb{D}_R)$, let $c_n(f) = f^{(n)}(0)/n!$. Choose $p \in [1,+\infty)$, and let μ denote the Lebesgue measure on the circle $|z| = r$. Show that the following families of seminorms on $\mathscr{O}(\mathbb{D}_R)$ are equivalent:

(1) $\|f\|_K = \sup\limits_{z\in K} |f(z)| \quad (K \subset U \text{ is a compact set})$;

(2) $\|f\|_r^{(1)} = \sum_{n=0}^\infty |c_n(f)| r^n \quad (0 < r < R)$;

(3) $\|f\|_r^{(p)} = \Big(\sum_{n=0}^\infty (|c_n(f)| r^n)^p\Big)^{1/p} \quad (0 < r < R)$;

(4) $\|f\|_r^\infty = \sup_{n \geqslant 0} |c_n(f)| r^n \qquad (0 < r < R)$;

(5) $\|f\|_r^I = \int_{|z|=r} |f(z)|\, d\mu(z) \qquad (0 < r < R)$;

(6) $\|f\|_r^{I,p} = \left(\int_{|z|=r} |f(z)|^p \, d\mu(z) \right)^{1/p} \qquad (0 < r < R)$.

3.5*. Let X be a finite-dimensional vector space. Show that there is only one topology on X which makes X into a Hausdorff topological vector space, and that this topology is determined by any norm on X. (This result was proved at the lectures in the special case of locally convex topologies.)

3.6*. Prove that a topological vector space is semimetrizable if and only if its topology is generated by an F-seminorm. (This result was proved at the lectures in the special case of locally convex spaces.)

3.7. Let S be an infinite set. Show that there are no continuous norms on $\mathbb{K}^S$. As a corollary, $\mathbb{K}^S$ is not normable.

3.8. Let X be a noncompact, completely regular (i.e., Tychonoff) topological space. Show that there are no continuous norms on $C(X)$. As a corollary, $C(X)$ is not normable.

3.9. Let $U \subset \mathbb{R}^n$ be a nonempty open set. Show that there are no continuous norms on $C^\infty(U)$. As a corollary, $C^\infty(U)$ is not normable.

3.10. Show that the following spaces are not normable, although each of them has a continuous norm: **(a)** s; **(b)** $C^\infty[a,b]$; **(c)** $\mathscr{S}(\mathbb{R}^n)$; **(d)** $\mathscr{O}(U)$ (where U is a domain in $\mathbb{C}$).

3.11. Prove that the following spaces are metrizable:
(1) $C(X)$, where X is a second countable, locally compact topological space;
(2) $C^\infty(U)$, where $U \subset \mathbb{R}^n$ is an open set.

3.12. Let S be a set. Show that $\mathbb{K}^S$ is metrizable if and only if S is at most countable.

3.13. Show that the strongest locally convex space is metrizable if and only if it is finite-dimensional.

3.14. Let X be a normed space. Show that
(a) the dual space X' equipped with the weak* topology is metrizable if and only if the dimension of X is at most countable;
(2) X equipped with the weak topology is metrizable if and only if it is finite-dimensional.