Convention. All vector spaces are over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

1.1. Let X be a topological vector space. Show that the closure of a vector subspace $X_0 \subset X$ is a vector subspace as well.

1.2. Let X and Y be topological vector spaces. Show that
(1) a linear map $X \to Y$ is continuous iff it is continuous at 0;
(2) the set $\mathscr{L}(X, Y)$ of all continuous linear maps from X to Y is a vector subspace of $\mathrm{Hom}_{\mathbb{K}}(X, Y)$.

1.3. Let (X, P) be a polynormed space. Show that the topology on X generated by P makes X into a topological vector space.
Hint: the shortest way is to reduce everything to seminormed spaces.

1.4. Let (X, P) be a polynormed space. Show that a sequence (x_n) in X converges to $x \in X$ w.r.t. the topology generated by P iff for all $p \in P$ we have $p(x_n - x) \to 0$.

1.5. Let (X, P) be a polynormed space. Show that $\overline{\{0\}} = \bigcap\{p^{-1}(0) : p \in P\}$.

1.6. Give a reasonable definition of the canonical topology on $C^\infty(M)$, where M is a smooth manifold. (This was done at the lecture in the special cases where M is either a closed interval on $\mathbb{R}$ or an open subset of $\mathbb{R}^n$.)

1.7. Let $U \subset \mathbb{C}$ be an open set. Show that the topology of compact convergence on the space $\mathscr{O}(U)$ of holomorphic functions is the same as the topology inherited from $C^\infty(U)$.

1.8. Let X be a vector space. Show that $S \subset X$ is convex iff for all $\lambda, \mu \geqslant 0$ we have $(\lambda + \mu)S = \lambda S + \mu S$.

1.9. Let X be a vector space, and let $S \subset X$. Show that
(1) $\mathrm{conv}(S) = \Big\{\sum_{i=1}^n \lambda_i x_i : x_i \in S,\ \lambda_i \geqslant 0,\ \sum_{i=1}^n \lambda_i = 1,\ n \in \mathbb{N}\Big\}$;
(2) $\mathrm{circ}(S) = \Big\{\lambda x : x \in S,\ \lambda \in \mathbb{K},\ |\lambda| \leqslant 1\Big\}$;
(3) $\Gamma(S) = \Big\{\sum_{i=1}^n \lambda_i x_i : x_i \in S,\ \lambda_i \in \mathbb{K},\ \sum_{i=1}^n |\lambda_i| \leqslant 1,\ n \in \mathbb{N}\Big\}$.

1.10. Let X be a topological vector space, and let $S \subset X$. Show that
(1) if S is convex, then the closure $\overline{S}$ and the interior $\mathrm{Int}\, S$ are convex;
(2) if S is circled, then $\overline{S}$ is circled; if, in addition, $0 \in \mathrm{Int}\, S$, then $\mathrm{Int}\, S$ is circled;
(3) if S is open, then $\mathrm{conv}(S)$ and $\Gamma(S)$ are open; if, in addition, $0 \in S$, then $\mathrm{circ}(S)$ is open.