K_1 for Banach algebras. The index map

(EXERCISES FOR LECTURES 16–19)

6.1. Given a (not necessarily unital) ring R, recall that the group $\mathrm{GL}_n^+(R)$ is defined to be the kernel of the homomorphism $\mathrm{GL}_n(R_+) \to \mathrm{GL}_n(\mathbb{Z})$ induced by the augmentation $R_+ \to \mathbb{Z}$.
(a) Suppose that R is a two-sided ideal of a unital ring S. Construct an isomorphism between $\mathrm{GL}_n^+(R)$ and $\mathrm{Ker}(\mathrm{GL}_n(S) \to \mathrm{GL}_n(S/R))$.
(b) Deduce that, if R is already unital, then $\mathrm{GL}_n^+(R) \cong \mathrm{GL}_n(R)$.

6.2. Let A be a separable Banach algebra. Without using the isomorphism $K_1(A) \cong K_0(SA)$, show that $K_1(A)$ is at most countable.

6.3. Let A be a Banach algebra. Equip $\mathrm{GL}_\infty(A)$ with the inductive (=final) topology generated by the inclusions $\mathrm{GL}_n(A) \hookrightarrow \mathrm{GL}_\infty(A)$ for all $n \in \mathbb{N}$. Show that
(a) $\mathrm{GL}_\infty(A)$ is a topological group, and $\mathrm{GL}_\infty(A) = \varinjlim \mathrm{GL}_n(A)$ in the category of topological groups;
(b) two elements of $\mathrm{GL}_\infty(A)$ are homotopic iff they are homotopic in $\mathrm{GL}_n(A)$ for some $n \in \mathbb{N}$;
(c) $\mathrm{GL}_\infty(A)$ is locally path connected;
(d) for every topological group G, the set $\pi_0(G)$ of path connected components of G is naturally isomorphic to the quotient G/G_0, where G_0 is the path connected component of the identity (show, in particular, that G_0 is a normal subgroup of G);
(e) there exists a natural isomorphism $K_1(A) \cong \pi_0(\mathrm{GL}_\infty(A))$.

6.4. Given a Banach algebra A, construct an isomorphism $K_1(A) \cong \varinjlim(\mathrm{GL}_n(A)/\mathrm{GL}_n(A)_0)$.

6.5. Without using the isomorphism $K_1(A) \cong K_0(SA)$, show that the functor K_1 defined on the category of Banach algebras is **(a)** half exact; **(b)** split exact; **(c)** homotopy invariant; **(d)** continuous; **(e)** stable; **(f)** satisfies $K_1(A \times B) \cong K_1(A) \times K_1(B)$; **(g)** satisfies $K_1(A^{\mathrm{op}}) \cong K_1(A)$.

6.6. Let A be a C^*-inductive limit of finite-dimensional C^*-algebras (see examples in Exercise Sheet 5). Show that $K_1(A) = 0$.

6.7. Calculate $K_1(C^*(\mathrm{F}_2))$. (*Hint:* see the hint to Exercise 4.15.)

6.8. Prove the naturality of the index map $\mathrm{GL}_\infty(S) \to K_0(I)$ associated to a ring extension $I \hookrightarrow R \to S$ (see the lectures for a precise statement).

6.9. Let $A \to B$ be a surjective $*$-homomorphism between C^*-algebras. Without using the Bartle-Graves theorem, show that, for every locally compact Hausdorff topological space X, the induced map $C_0(X, A) \to C_0(X, B)$ is surjective. Deduce that the functor $C_0(X, -)$ takes C^*-algebra extensions to C^*-algebra extensions.

6.10. Given a pointed topological space (Y, y_0), let ΣY denote the (topological) reduced suspension over Y, i.e., the space obtained from $Y \times [0,1]$ by collapsing $(Y \times \{0\}) \cup (Y \times \{1\}) \cup (\{y_0\} \times [0,1])$ to a point v_0. Show that, if X is a locally compact Hausdorff topological space, then the (C^*-algebraic) suspension over $C_0(X)$ is isomorphic to $C_0(\Sigma(X_+) \setminus \{v_0\})$.

6.11. Let G be a topological group, and let $[S^1, G]$ denote the set of homotopy classes of continuous maps from S^1 to G.
(a) Show that $[S^1, G]$ is the quotient of $C(S^1, G)$ modulo a certain normal subgroup and hence it is itself a group in a natural way. (Here the multiplication on $C(S^1, G)$ comes from the multiplication on G.)
(b) Let $G_0 \subset G$ denote the path connected component of the identity $e \in G$, and let $E = \{[f] \in [S^1, G] : f(1) \in G_0\}$. Show that E is a subgroup of $[S^1, G]$, and construct an isomorphism between E and the fundamental group $\pi_1(G, e)$.
(c) Given a Banach algebra A, construct an isomorphism $K_2(A) \cong \pi_1(\mathrm{GL}_\infty(A), e)$. (*Hint:* use the split extension $SA \hookrightarrow C(S^1, A) \to A$.)