Inductive limits of C^*-algebras. Continuity and stability of K_0

(EXERCISES FOR LECTURES 13–15)

5.1. (a) Let A be a set, and let $(A_i)_{i\in I}$ be a family of subsets of A indexed by a directed poset I in such a way that $A_i \subset A_j$ whenever $i \leqslant j$. Thus the sets A_i together with the inclusion maps $A_i \hookrightarrow A_j$ form an inductive system. Show that $\varinjlim A_i = \bigcup_i A_i$, and that the canonical maps $A_i \to \varinjlim A_i$ are the inclusion maps.
(b) Prove a similar result for the categories of groups, R-modules (where R is a ring), k-algebras (where k is a commutative ring), $*$-algebras.
(c) Prove a similar result for the categories Ban_1 (objects = Banach spaces, morphisms = contractive linear maps), BanAlg_1 (objects = Banach algebras, morphisms = contractive algebra homomorphisms), C^*-Alg (objects = C^*-algebras, morphisms = $*$-homomorphisms). In all these cases, we have $\varinjlim A_i = \overline{\bigcup_i A_i}$.

5.2. Let $E_1 \subset E_2 \subset E_3 \subset \ldots$ be an increasing sequence of closed vector subspaces of a Banach space E. Show that such a sequence does not have an inductive limit in the category of Banach spaces and bounded linear maps unless it stabilizes.

5.3. Verify the details of the construction of $\varinjlim A_i$ given at the lecture for the categories of
(a) sets; **(b)** abelian groups; **(c)** rings; **(d)** $*$-algebras; **(e)** C^*-algebras.

5.4. Let (A_i) be an inductive system of C^*-algebras. Construct a natural isomorphism between $C^*\text{-}\varinjlim((A_i)_+)$ and $(C^*\text{-}\varinjlim A_i)_+$.

5.5. Let $A = C^*\text{-}\varinjlim M_n$, where $M_n = M_n(\mathbb{C})$ is the algebra of complex $n \times n$-matrices, and the maps $M_n \hookrightarrow M_{n+1}$ are given by $a \mapsto a \oplus 0$. Show that $A \cong \mathcal{K}$, the algebra of compact operators on an infinite-dimensional separable Hilbert space.

Hint. Choose an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H, let P_n denote the projection onto $\operatorname{span}\{e_1, \ldots, e_n\}$, and let $A_n = P_n\mathcal{K}P_n$. Show that (A_n) and (M_n) are isomorphic as inductive sequences of C^*-algebras, and that $\bigcup_n A_n$ is dense in $\mathcal{K}$.

5.6. Given $n \in \mathbb{N}$, define a homomorphism $M_n \oplus \mathbb{C} \to M_{n+1} \oplus \mathbb{C}$ by $(a, \lambda) \mapsto (a \oplus \lambda, \lambda)$. Identify the C^*-inductive limit A of this sequence, and calculate $K_0(A)$. (*Hint*: A is a certain operator C^*-algebra related to $\mathcal{K}$.)

5.7. Define $d\colon \mathbb{C} \to \mathbb{C} \oplus \mathbb{C}$ by $d(z) = (z, z)$, and consider the inductive sequence

$$\mathbb{C} \xrightarrow{d} \mathbb{C}^2 \xrightarrow{d\oplus d} \mathbb{C}^4 \xrightarrow{d^{\oplus 4}} \mathbb{C}^8 \xrightarrow{d^{\oplus 8}} \ldots$$

of C^*-algebras.
(a) Show that $C^*\text{-}\varinjlim \mathbb{C}^{2^n} \cong C(K)$, the algebra of continuous functions on the Cantor set.
(b) Construct an isomorphism $K_0(C(K)) \cong C(K, \mathbb{Z})$.

5.8. Given $n \in \mathbb{N}$, define a homomorphism $M_{2^n} \to M_{2^{n+1}}$ by $a \mapsto a \oplus a$. The C^*-algebra $A = C^*\text{-}\varinjlim M_{2^n}$ is called the *CAR algebra* (CAR means "canonical anticommutation relations"). Show that $K_0(A) \cong \mathbb{Z}[1/2]$, the group of dyadic rationals.

5.9. Given $n \in \mathbb{N}$, define a homomorphism $M_{2^n} \oplus M_{2^n} \to M_{2^{n+1}} \oplus M_{2^{n+1}}$ by $(a, b) \mapsto (a \oplus b, a \oplus b)$. Show that the C^*-inductive limit of this sequence is isomorphic to the CAR algebra (see Exercise 5.8).

5.10. Let $A = C^*\text{-}\varinjlim M_{n!}$, where the maps $M_{n!} \to M_{(n+1)!}$ are given by $a \mapsto \underbrace{a \oplus \cdots \oplus a}_{n+1}$.

Calculate $K_0(A)$.

5.11. Define sequences (m_k) and (n_k) recursively by

$$m_1 = n_1 = 1, \quad m_{k+1} = m_k + n_k, \quad n_{k+1} = m_k$$

(thus both (m_k) and (n_k) are the Fibonacci numbers shifted by 1). Let $A_k = M_{m_k} \oplus M_{n_k}$, and define a homomorphism $A_k \to A_{k+1}$ by $(a, b) \mapsto (a \oplus b, a)$. The *Fibonacci algebra* is defined by $A = C^*\text{-}\varinjlim A_k$. Calculate $K_0(A)$.

5.12. Given $n \in \mathbb{N}$, let $A_n = \bigoplus_{k=0}^{n} M_{\binom{n}{k}}$, and define a homomorphism $A_n \to A_{n+1}$ by

$$(a_1, \ldots, a_n) \mapsto (a_1, a_1 \oplus a_2, a_2 \oplus a_3, \ldots, a_{n-1} \oplus a_n, a_n).$$

The C^*-algebra $A = C^*\text{-}\varinjlim A_n$ is called the *GICAR algebra* (GICAR means "gauge invariant canonical anticommutation relations"). Show that

$$K_0(A) \cong \{(x+1)^{-n} p(x) : p \in \mathbb{Z}[x],\ \deg p \leqslant n,\ n \in \mathbb{Z}_{\geqslant 0}\}.$$

5.13. **(a)** Given $a \in M_{n,m}(\mathbb{Z}_{\geqslant 0})$, define $\ell_a \colon \mathbb{Z}^m \to \mathbb{Z}^n$ by $\ell_a(x) = ax$ (where x is viewed as a column vector). Construct finite-dimensional C^*-algebras A and B such that $K_0(A) \cong \mathbb{Z}^m$, $K_0(B) \cong \mathbb{Z}^n$, and $\ell_a = (\varphi_a)_*$ for a suitable $*$-homomorphism $\varphi_a \colon A \to B$.
(b) Show that for every countable torsion-free abelian group G there exists a C^*-algebra A such that $K_0(A) \cong G$.

Hints. **(a)** $A = M_{k_1} \oplus \cdots \oplus M_{k_m}$ and $B = M_{\ell_1} \oplus \cdots \oplus M_{\ell_n}$, where $k_1, \ldots k_m$ are arbitrary and $\ell_1, \ldots, \ell_n$ are big enough. **(b)** Write G as an inductive limit of free finitely generated abelian groups and connecting maps as in **(a)**.

5.14. Given $n \in \mathbb{N}$, let $A_n = C[-1/n, 1/n]$, and let $A_n \to A_{n+1}$ be the restriction map. Let also $B_n = \{f \in A_n : f(0) = 0\}$. Show that
(a) $A = \operatorname{alg}\varinjlim A_n$ is the algebra of germs of continuous functions at $0 \in \mathbb{R}$, $B = \operatorname{alg}\varinjlim B_n$ is the maximal ideal of A consisting of germs vanishing at 0, and both A and B are infinite-dimensional $*$-algebras (here $\operatorname{alg}\varinjlim$ denotes the inductive limit in the category of $*$-algebras);
(b) $C^*\text{-}\varinjlim A_n \cong \mathbb{C}$ and $C^*\text{-}\varinjlim B_n = 0$.

5.15. Given a C^*-algebra A, construct a natural homomorphism $K_{00}(A) \to K_0(A)$ (where $K_{00}(A)$ is defined in Exercise 2.4), and show that this map is an isomorphism provided that A has a countable approximate identity consisting of projections.

(*Hint:* see the hint to Exercise 5.5.)

5.16. Let A be a C^*-algebra. Show that
(a) for each pair $p, q \in \Pr(A \otimes_* \mathcal{K})$ there exist $p', q' \in \Pr(A \otimes_* \mathcal{K})$ such that $p' \underset{\text{MvN}}{\sim} p$, $q' \underset{\text{MvN}}{\sim} q$, and $p' \perp q'$;
(b) $\Pr(A \otimes_* \mathcal{K})$ is an abelian semigroup under the operation $[p] + [q] = [p' + q']$ (show, in particular, that the operation is well defined);
(c) if A is unital, then $K_0(A) \cong \operatorname{Gr}(\Pr(A \otimes_* \mathcal{K}))$.