K_0 for C^*-algebras. Projections and unitaries

(EXERCISES FOR LECTURES 12–13)

4.1. Let H be a Hilbert space, and let A be a $*$-subalgebra of $\mathscr{B}(H)$ containing $\mathbf{1}_H$. Show that
(a) $u \in A$ is an isometry (i.e., $u^*u = 1$) $\iff$ $\|u(x)\| = \|x\|$ for all $x \in H$ $\iff$ $\langle u(x) \,|\, u(y)\rangle = \langle x \,|\, y\rangle$ for all $x, y \in H$;
(b) $u \in A$ is unitary $\iff$ u is a surjective (or, equivalently, bijective) isometry;
(c) $u \in A$ is a coisometry (i.e., $uu^* = 1$) $\iff$ there exists a closed vector subspace $H_0 \subset H$ such that $u|_{H_0^\perp} = 0$ and that $u|_{H_0}$ maps H_0 isometrically onto H (in fact, $H_0 = (\operatorname{Ker} u)^\perp$).

4.2. Let A be a unital C^*-algebra, $A \neq 0$. Show that
(a) if $u \in A$ is an isometry or a coisometry, then $\|u\| = 1$;
(b) if $u \in A$ is unitary, then $\sigma(u) \subset \mathbb{T}$ (where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$);
(c) if $u \in A$ is a noninvertible isometry or a noninvertible coisometry, then $\sigma(u) = \bar{\mathbb{D}}$ (where $\bar{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leqslant 1\}$).

4.3. Let H be a Hilbert space, and let A be a $*$-subalgebra of $\mathscr{B}(H)$. Show that
(a) $p \in A$ is a projection (i.e., $p = p^2 = p^*$) $\iff$ there exists a closed vector subspace $H_0 \subset H$ such that $p|_{H_0} = \mathbf{1}_{H_0}$ and $p|_{H_0^\perp} = 0$ (in fact, $H_0 = \operatorname{Im} p = (\operatorname{Ker} p)^\perp$);
(b) $v \in A$ is a partial isometry (i.e., $vv^*v = v$) $\iff$ there exist closed vector subspaces $H_0, H_1 \subset H$ such that $v|_{H_0^\perp} = 0$ and that $v|_{H_0}$ maps H_0 isometrically onto H_1 (in fact, $H_0 = (\operatorname{Ker} v)^\perp = \operatorname{Im} v^*$ and $H_1 = \operatorname{Im} v = (\operatorname{Ker} v^*)^\perp$). How does v^* act on H_1 and on $(H_1)^\perp$?

4.4. Let A be a C^*-algebra. Show that
(a) if $v \in A$ is a partial isometry, then $\|v\| \in \{0, 1\}$;
(b) if $e \in \operatorname{Idem} A$, then e is a projection iff $\|e\| \in \{0, 1\}$.

4.5. Construct two similar idempotents in $M_2(\mathbb{C})$ which are not unitarily equivalent.

4.6. Given a $*$-algebra A, show that the Murray-von Neumann equivalence is an equivalence relation on $\Pr(A)$.

4.7. Given a $*$-algebra A and projections $p, q \in A$, show that p and q are Murray-von Neumann equivalent in A iff they are Murray-von Neumann equivalent in A_+.

4.8. Let A be a unital C^*-algebra, and let $p, q \in \Pr A$. We say that p and q are *unitopic* if there is a continuous path $t \mapsto u_t$ in $\mathrm{U}(A)$ such that $u_0 = 1$ and $u_1 p u_1^{-1} = q$. Show that
(a) unitopy defines an equivalence relation on $\Pr A$;
(b) if $\|p - q\| < 1$, then p and q are unitopic (and hence are unitarily equivalent);
(c) p and q are unitopic iff they are homotopic.
(*Hint:* reduce (b), (c) to similar statements about simeotopy, see the lectures.)

4.9. Show that a retract of a locally path connected topological space is locally path connected. Deduce that, for each C^*-algebra A, the spaces $\Pr(A)$ and (if A is unital) $\mathrm{U}(A)$ are locally path connected.

4.10. Let A be a unital C^*-algebra. Given an isometry $u \in A$, define a $*$-homomorphism $\alpha_u \colon A \to A$ by $\alpha_u(a) = uau^*$. Show that α_u induces the identity map on $K_0(A)$.

4.11. Construct a unital C^*-algebra A and a $*$-automorphism α of A such that $\alpha_*\colon K_0(A) \to K_0(A)$ is not the identity map.

4.12. Let H be an infinite-dimensional separable Hilbert space, and let $\mathcal{Q}(H) = \mathscr{B}(H)/\mathcal{K}(H)$ be the Calkin algebra. Show that $K_0(\mathcal{Q}(H)) = 0$.

Hint. Prove that every projection in $\mathcal{Q}(H)$ lifts to a projection in $\mathscr{B}(H)$. Towards this goal, lift a projection $p \in \mathcal{Q}(H)$ to a selfadjoint operator $T \in \mathscr{B}(H)$ and, by using the Hilbert-Schmidt decomposition of the compact selfadjoint operator $T - T^2$, construct explicitly a compact selfadjoint operator K on H such that $T + K$ is a projection.

Let G be a group. A (full) *group C^*-algebra* of G is a unital C^*-algebra $C^*(G)$ together with a natural isomorphism

$$\mathrm{Hom}_{\mathsf{Un.}C^*\text{-}\mathsf{Alg}}(C^*(G), A) \cong \mathrm{Hom}_{\mathsf{Groups}}(G, \mathrm{U}(A))$$

where $\mathsf{Un.}C^*\text{-}\mathsf{Alg}$ is the category of unital C^*-algebras and Groups is the category of groups. More explicitly, $C^*(G)$ is equipped with a group homomorphism $i\colon G \to \mathrm{U}(C^*(G))$ such that for each unital C^*-algebra A and each group homomorphism $\varphi\colon G \to \mathrm{U}(A)$ there exists a unique unital $*$-homomorphism $\psi\colon C^*(G) \to A$ satisfying $\psi \circ i = \varphi$.

4.13. Let $\mathbb{C}G$ be the (algebraic) group algebra of G. For each $a \in \mathbb{C}G$, let

$$\|a\|_{\max} = \sup\bigl\{ \|\pi(a)\| : \pi \text{ is a } *\text{-representation of } \mathbb{C}G \bigr\}$$

Show that

(a) $\|a\|_{\max} < \infty$ for each $a \in \mathbb{C}G$;

(b) $\|\cdot\|_{\max}$ is a C^*-norm on $\mathbb{C}G$;

(c) for each C^*-seminorm p on $\mathbb{C}G$, we have $p \leqslant \|\cdot\|_{\max}$;

(d) $C^*(G)$ exists and is the completion of $\mathbb{C}G$ w.r.t. $\|\cdot\|_{\max}$.

4.14. Show that $C^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$.

4.15. Let F_2 be the free group on two generators, and let u_1, u_2 denote the unitary elements of $C^*(\mathrm{F}_2)$ corresponding to the free generators of F_2. Let also v_1, v_2 denote the unitary elements of $C(S^1 \vee S^1)$ corresponding to the canonical maps $S^1 \vee S^1 \to S^1$. By the universal property of $C^*(\mathrm{F}_2)$ (see above), there exists a unique $*$-homomorphism $\alpha\colon C^*(\mathrm{F}_2) \to C(S^1 \vee S^1)$ such that $\alpha(u_i) = v_i$ $(i = 1, 2)$.

(a) Show that $\alpha_*\colon K_0(C^*(\mathrm{F}_2)) \to K_0(C(S^1 \vee S^1))$ is an isomorphism.

(b) Calculate $K_0(C^*(\mathrm{F}_2))$. (You may use the fact that $K^0(S^1) \cong \mathbb{Z}$.)

Hints. Let $A = C^*(\mathrm{F}_2)$ and $B = C(S^1 \vee S^1)$.

(i) Using Exercise 4.14, find a universal property of B in terms of v_1, v_2.

(ii) Show that there exists a $*$-homomorphism $\beta\colon B \to M_2(A)$ uniquely determined by $\beta(v_1) = u_1 \oplus 1$ and $\beta(v_2) = 1 \oplus u_2$.

(iii) Let $\eta_A\colon \mathbb{C} \to A$ be the (unique) unital homomorphism from $\mathbb{C}$ to A, and let $\varepsilon_A\colon A \to \mathbb{C}$ be the $*$-homomorphism uniquely determined by $u_i \mapsto 1$ $(i = 1, 2)$. Construct an homotopy between $\beta\alpha$ and a $*$-homomorphism $\gamma\colon A \to M_2(A)$ such that, after identifying $K_0(M_2(A))$ with $K_0(A)$, we have $\gamma_* = \mathbf{1} + (\eta_A\varepsilon_A)_*\colon K_0(A) \to K_0(A)$.

(iv) Denote the canonical extension of α to a map $M_2(A) \to M_2(B)$ by the same letter α. As in (ii), construct an homotopy between $\alpha\beta$ and a $*$-homomorphism $\delta\colon B \to M_2(B)$ such that $\delta_* = \mathbf{1} + (\eta_B\varepsilon_B)_*\colon K_0(B) \to K_0(B)$, where ε_B is defined similarly to ε_A (show that it exists).

(v) Show that $\beta_* - (\eta_A\varepsilon_B)_*\colon K_0(B) \to K_0(A)$ is inverse to α_*.