K_0 for Banach algebras. Homotopy invariance

(EXERCISES FOR LECTURES 11–12)

3.1. Let A be a unital Banach algebra, and let $e, f \in \operatorname{Idem} A$. We say that e and f are *simeotopic* if there is a continuous path $t \mapsto u_t$ in $A^\times$ such that $u_0 = 1$ and $u_1 e u_1^{-1} = f$. Show that simeotopy defines an equivalence relation on $\operatorname{Idem} A$. (In fact, this relation is equivalent to homotopy, see the lectures.)

3.2. Let A be a Banach algebra. Show that $\operatorname{Idem} A$ is locally path connected (i.e., for each $e \in \operatorname{Idem} A$ and each open neighborhood U of e there is a path connected open neighborhood V of e such that $V \subset U$).

3.3. Let A be a unital Banach algebra, and let $x, y \in A^\times$. Show that $x \oplus y$ and $y \oplus x$ are homotopic in $\mathrm{GL}_2(A)$. (This is a part of Whitehead's lemma, see the lectures.)

3.4. Let A be a separable Banach algebra. Show that $K_0(A)$ is at most countable.

3.5. Let A and B be Banach algebras, and let $\operatorname{Hom}(A, B)$ denote the set of continuous homomorphisms from A to B equipped with the topology of pointwise convergence (i.e., the topology induced from B^A).
(a) Construct a natural bijection

$$\operatorname{Hom}(A, C([0,1], B)) \cong C([0,1], \operatorname{Hom}(A, B)).$$

(b) Given $\varphi_0, \varphi_1 \in \operatorname{Hom}(A, B)$, show that φ_0 and φ_1 are homotopic as Banach algebra homomorphisms (i.e., there exists a continuous homomorphism $\Phi\colon A \to C([0,1], B)$ such that $\mathrm{ev}_k \circ \Phi = \varphi_k$ for $k = 0, 1$) if and only if they are homotopic as elements of $\operatorname{Hom}(A, B)$ (i.e., belong to the same path connected component of $\operatorname{Hom}(A, B)$).

3.6. Let X and Y be compact Hausdorff topological spaces, $f_0, f_1\colon X \to Y$ be continuous maps, and $f_0^\bullet, f_1^\bullet\colon C(Y) \to C(X)$ be the homomorphisms induced by f_0 and f_1. Show that
(a) f_0 and f_1 are homotopic (as continuous maps) iff $f_0^\bullet$ and $f_1^\bullet$ are homotopic as Banach algebra homomorphisms;
(b) f_0 is a homotopy equivalence iff $f_0^\bullet$ is a homotopy equivalence.

3.7. Let X be a compact Hausdorff topological space, and let $x_0 \in X$. When is $C_0(X \setminus \{x_0\})$ contractible? Give a criterion entirely in terms of X and x_0.

3.8. Given a compact Hausdorff topological space X, let $\operatorname{cone}(X)$ denote the (topological) cone over X, i.e., the space obtained from $X \times [0,1]$ by collapsing $X \times \{0\}$ to a point (which will be denoted by v_0). Construct a C^*-algebra isomorphism between the cone over $C(X)$ and $C_0(\operatorname{cone}(X) \setminus \{v_0\})$.

Given a Banach algebra A, the *suspension* over A is $SA = C_0((0,1), A)$ (or, equivalently, $C_0(\mathbb{R}, A)$). Given a Banach algebra homomorphism $\varphi\colon A \to B$, the *mapping cylinder* and the *mapping cone* of φ are defined, respectively, by

$$Z_\varphi = \{(a, f) \in A \oplus C([0,1], B) : \varphi(a) = f(1)\}, \qquad C_\varphi = \{(a, f) \in A \oplus C_0((0,1], B) : \varphi(a) = f(1)\}.$$

3.9. Given a compact Hausdorff topological space X, interpret the suspension over $C(X)$ in terms of the (topological) suspension $\Sigma X = \operatorname{cone}(X)/(X \times \{1\})$.

3.10. Let X and Y be compact Hausdorff topological spaces, $f: X \to Y$ be a continuous map, and $\varphi = f^\bullet: C(Y) \to C(X)$ be the homomorphism induced by f. Interpret Z_φ and C_φ in terms of the (topological) mapping cylinder and the mapping cone defined, respectively, by

$$\operatorname{cyl}(f) = (X \times [0,1]) \cup_g Y, \qquad \operatorname{cone}(f) = \operatorname{cone}(X) \cup_g Y,$$

where $g: X \times \{1\} \to Y$ is given by $g(x,1) = f(x)$.

3.11. Given a Banach algebra A, determine the mapping cylinder and the mapping cone of the homomorphisms $\mathbf{1}_A: A \to A$ and $0 \to A$ (i.e., characterize them in terms of some standard constructions applied to A).

3.12. Show that a Banach algebra homomorphism $\varphi: A \to B$ is homotopic to the zero map iff it lifts to CB (i.e., there exists a continuous homomorphism $\Phi: A \to CB$ such that $\mathrm{ev}_1 \circ \Phi = \varphi$).

3.13. Let $\varphi: A \to B$ be a Banach algebra homomorphism, and let p denote the projection of C_φ onto A (i.e., $p(a,f) = a$ for all $(a,f) \in C_\varphi$). Show that
(a) A is homotopy equivalent to Z_φ;
(b) the sequence $K_0(C_\varphi) \xrightarrow{p_*} K_0(A) \xrightarrow{\varphi_*} K_0(B)$ is exact.

3.14. Let $\varphi: A \to B$ be a surjective C^*-algebra homomorphism, and let $I = \operatorname{Ker}\varphi$. Construct an isomorphism $K_0(I) \cong K_0(C_\varphi)$.

Hint. Use an embedding $\beta: I \hookrightarrow C_\varphi$, show that C_φ/I is contractible, and construct an extension $D \hookrightarrow C_\beta \to E$ such that D and E are contractible.