K_0 for unital rings

(EXERCISES FOR LECTURES 7–9)

1.1. Let S be an abelian semigroup. Verify the details of the construction of the Grothendieck group $\mathrm{Gr}(S)$ given at the lecture. Specifically, show that
(a) the following two definitions yield the same equivalence relation on $S \times S$:
- **(1)** $(x_1, y_1) \sim (x_2, y_2)$ iff $(x_1 + z_1, y_1 + z_1) = (x_2 + z_2, y_2 + z_2)$ for some $z_1, z_2 \in S$;
- **(2)** $(x_1, y_1) \sim (x_2, y_2)$ iff $x_1 + y_2 + z = x_2 + y_1 + z$ for some $z \in S$;

(b) the quotient $G = (S \times S)/\sim$ becomes an abelian group under the operation $[(x_1, y_1)] + [(x_2, y_2)] = [(x_1 + x_2, y_1 + y_2)]$ (where $[(x, y)]$ stands for the equivalence class of $(x, y) \in S \times S$);
(c) the map $i\colon S \to G$ given by $i(x) = [(x + x, x)]$ is a semigroup homomorphism, and it has the universal property that, for each abelian group H and each semigroup homomorphism $f\colon S \to H$, there is a unique group homomorphism $g\colon G \to H$ such that $g \circ i = f$;
(d) for all $x, y \in S$ we have $[(x, y)] = i(x) - i(y)$;
(e) given $x, y \in S$, we have $i(x) = i(y)$ iff $x + z = y + z$ for some $z \in R$.

1.2. Let S_1 and S_2 be abelian semigroups. Show that the projections $p_k\colon S_1 \times S_2 \to S_k$ $(k = 1, 2)$ induce a group isomorphism $\mathrm{Gr}(S_1 \times S_2) \xrightarrow{\sim} \mathrm{Gr}(S_1) \times \mathrm{Gr}(S_2)$ given by $a \mapsto (p_{1,*}(a), p_{2,*}(a))$.

1.3. Calculate $\mathrm{Gr}(S)$ for the following semigroups S (in particular, describe the canonical map $i\colon S \to \mathrm{Gr}(S)$ explicitly): **(a)** $S = \mathbb{Z}_{\geqslant 0}$; **(b)** $S = \mathbb{Z}_{\geqslant 1}$; **(c)** $S = \mathbb{Z}_{\geqslant 2}$; **(d)** $S = (\mathbb{Z} \setminus \{0\}, \cdot)$; **(e)** $S = \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ with operation $+$ extended to S from $\mathbb{Z}_{\geqslant 0}$ by $n + \infty = \infty$ for all n; **(f)** $S = \{0, x, 2x, \ldots, (n-1)x\}$ with obvious operation $+$ and relation $nx = x$.

1.4. Let F be a free finitely generated module over a unital ring. Show that each basis of F is finite.

1.5. Show that a finitely generated abelian group is a projective $\mathbb{Z}$-module iff it is free.

1.6. Let R_1 and R_2 be unital rings.
(a) Show that the projections $p_k\colon R_1 \times R_2 \to R_k$ $(k = 1, 2)$ induce a semigroup isomorphism $\mathbb{V}(R_1 \times R_2) \xrightarrow{\sim} \mathbb{V}(R_1) \times \mathbb{V}(R_2)$ given by $s \mapsto (p_{1,*}(s), p_{2,*}(s))$.
(b) Construct a similar isomorphism $K_0(R_1 \times R_2) \xrightarrow{\sim} K_0(R_1) \times K_0(R_2)$.

1.7. Let R be a unital ring, and let R-fgp (respectively, fgp-R) denote the category of left (respectively, right) finitely generated projective R-modules.
(a) Show that the functor $\mathrm{Hom}_R(-, R)$ is an equivalence between R-fgp and $(\mathsf{fgp}\text{-}R)^{\mathrm{op}}$.
(b) Deduce that $\mathbb{V}(R) \cong \mathbb{V}(R^{\mathrm{op}})$.

1.8. Let R be a unital ring, and let S be the semigroup of isomorphism classes of *countably* generated projective R-modules. Show that $\mathrm{Gr}(S) = 0$.

1.9 (*A geometric model for equivalence of idempotents*)**.** Let R be a subring of $\mathrm{End}_k(V)$, where V is a vector space over a field k. Suppose e, f are idempotents of R, and let $V_1 = \mathrm{Im}(e)$, $V_2 = \mathrm{Ker}(e)$, $W_1 = \mathrm{Im}(f)$, $W_2 = \mathrm{Ker}(f)$ (thus we have direct sum decompositions $V = V_1 \oplus V_2 = W_1 \oplus W_2$). Given $a, b \in R$, show that
(a) the pair (a, b) implements an equivalence between e and f (i.e., $ab = e$, $ba = f$) if and only if the following conditions hold:

$$b(V_1) = W_1, \quad a(W_1) = V_1, \quad ab|_{V_1} = \mathbf{1}_{V_1}, \quad ba|_{W_1} = \mathbf{1}_{W_1}, \tag{1}$$

$$b(V_2) \subset \mathrm{Ker}(a), \quad a(W_2) \subset \mathrm{Ker}(b). \tag{2}$$

(b) the pair (a, b) implements an equivalence between e and f (i.e., $ab = e$, $ba = f$) and is reduced (i.e., $ea = a = af$, $fb = b = be$) if and only if it satisfies (1) and $b(V_2) = a(W_2) = 0$.
(c) Assuming that $R = \mathrm{End}_k(V)$, show that $e \sim_a f$ iff $\dim \mathrm{Im}(e) = \dim \mathrm{Im}(f)$.

1.10. Let e, f be idempotents in a ring R.
(a) Suppose that $ef = fe = 0$ (in this case, we say that e and f are *orthogonal* and write $e \perp f$). Show that $e + f$ is an idempotent.
(b) Conversely, assume that R is a k-algebra (where $\mathrm{char}\, k \neq 2$) and that $e + f$ is an idempotent. Show that $e \perp f$.

1.11. Let R be a ring, and let M be an R-module. Suppose that p, q are idempotent endomorphisms of M such that $p \perp q$. Show that $\mathrm{Im}(p + q) = \mathrm{Im}(p) \oplus \mathrm{Im}(q)$.

1.12. Given a unital ring R, let $\tilde{\mathbb{V}}(R)$ denote the set of equivalence classes of idempotents in $M_\infty(R)$. Each unital ring homomorphism $\varphi \colon R \to S$ extends to a homomorphism $M_\infty(R) \to M_\infty(S)$, $(a_{ij}) \mapsto (\varphi(a_{ij}))$, which preserves equivalence of idempotents and hence induces a map $\tilde{\mathbb{V}}(R) \to \tilde{\mathbb{V}}(S)$. Thus $\tilde{\mathbb{V}}(R)$ becomes a functor on the category of unital rings. Show that the isomorphism $\tilde{\mathbb{V}}(R) \xrightarrow{\sim} \mathbb{V}(R)$, $[e] \mapsto [R^\infty e]$ (see the lecture) is natural in R.

1.13. Let R be a unital ring, and let $\tilde{\mathbb{V}}(R)$ be as in the previous exercise. Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{V}(R) & \longrightarrow & \mathbb{V}(R^{\mathrm{op}}) \\ \uparrow & & \uparrow \\ \tilde{\mathbb{V}}(R) & \longrightarrow & \tilde{\mathbb{V}}(R^{\mathrm{op}}) \end{array}$$

where the upper horizontal arrow is the isomorphism described in Exercise 1.7, and the vertical arrows are the natural isomorphisms described in the previous exercise. Explain (entirely in terms of matrices) how the lower horizontal arrow acts.

1.14. Calculate $K_0(R)$ for the following rings:
(a) $R = \mathbb{Z}/6\mathbb{Z}$; **(b)** $R = \mathbb{Z}/4\mathbb{Z}$; **(c)** $R = k[\varepsilon]/(\varepsilon^2)$ (k = a field).
(*Hint to* (b,c): there is an ideal $I \subset R$ such that $I^2 = 0$ and R/I is a field.)

1.15. Let R be a commutative unital ring. Show that, if P and Q are finitely generated projective R-modules, then so is $P \otimes_R Q$. Deduce that the tensor product operation makes $K_0(R)$ into a commutative unital ring.

1.16 (*Leavitt algebra)***.** Let k be a field. Given an integer $n \geqslant 2$, let R be the unital k-algebra generated by $2n$ elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ with relations

$$x_1 y_1 + \cdots x_n y_n = 1, \quad y_i x_j = \delta_{ij} 1 \quad (i, j = 1, \ldots, n). \tag{3}$$

(a) Let V be a k-vector space of countable dimension. Show that there exist linear operators $x_1, \ldots, x_n, y_1, \ldots, y_n$ on V satisfying (3). This implies, in particular, that $R \neq 0$. (*Hint:* use a decomposition $V = V_1 \oplus \cdots \oplus V_n$ with $V_i \cong V$.)
(b) Show that $R \cong R^n$ as R-modules. Deduce that $K_0(R)$ has torsion.

Remark 1.1. In fact, R is simple, so it is isomorphic to the subalgebra of $\mathrm{End}_k(V)$ generated by the operators from (a). Also, $K_0(R) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ (both facts are nontrivial).