Harmonic analysis on locally compact abelian groups

(EXERCISES FOR LECTURES 13, 15, 16)

Convention. Throughout, all locally compact groups are assumed to be second countable.

9.1. Let G be a nondiscrete locally compact abelian group. Prove that the embedding of $\hat{G}$ into $\operatorname{Max} M(G)$ given by $\chi \mapsto \tilde{\chi}$, where $\tilde{\chi}(\nu) = \hat{\nu}(\chi)$ (cf. Exercise 6.18 (b)), is not onto.

9.2. Let G be a locally compact abelian group. Show that
(a) G is compact iff $\widehat{G}$ is discrete;
(b) G is discrete iff $\widehat{G}$ is compact.

9.3. Construct an isomorphism between $\widehat{\mathbb{Z}}_p$ (see Exercise 4.11) and a certain subgroup of $\mathbb{T}$. Is the canonical topology on $\widehat{\mathbb{Z}}_p$ the same as the topology induced from $\mathbb{T}$?

9.4. (a) Given $x = \sum_k a_k p^k \in \mathbb{Q}_p$ (see Exercise 4.11), let $\lambda(x) = \sum_{k<0} a_k p^k \in \mathbb{Q}$. Define a character $\chi_x \colon \mathbb{Q}_p \to \mathbb{T}$ by $\chi_x(y) = e^{2\pi i \lambda(xy)}$. Show that χ_x is continuous, and that the map $\mathbb{Q}_p \to \widehat{\mathbb{Q}}_p$, $x \mapsto \chi_x$, is a topological isomorphism.
(b) Let μ denote the Haar measure on $\mathbb{Q}_p$ normalized in such a way that $\mu(\mathbb{Z}_p) = 1$ (see Exercise 4.12). Identify $\widehat{\mathbb{Q}}_p$ with $\mathbb{Q}_p$, as in (a). Find the respective Plancherel measure on $\mathbb{Q}_p$.

9.5. Let G be a compact abelian group equipped with the normalized Haar measure (i.e., the measure of G is 1). Show that the Plancherel measure on $\widehat{G}$ is the counting measure.

9.6. Let (G_i) be a family of compact abelian groups. Construct a topological isomorphism $(\prod G_i)\hat{} \cong \bigoplus \widehat{G}_i$ (where $\bigoplus \widehat{G}_i$ is equipped with the discrete topology).

9.7. Let $G = \mathbb{Q}/\mathbb{Z}$ equipped with the discrete topology. Construct a topological isomorphism between $\widehat{G}$ and $\prod_p \mathbb{Z}_p$.

A topological space X is *totally disconnected* if each nonempty connected subset of X is a singleton. (Standard examples: discrete spaces; $\mathbb{Q}$; the Cantor set and (more generally) arbitrary products of discrete spaces; $\mathbb{Z}_p$ and $\mathbb{Q}_p$.) It is known that a locally compact Hausdorff space X is totally disconnected iff the topology on X has a base consisting of clopen (i.e., closed and open) sets. You may use this fact below.

9.8. Let G be a compact abelian group. Show that
(a) G is connected iff $\widehat{G}$ is torsion-free (i.e., the only element of finite order in $\widehat{G}$ is the identity).
(b) G is totally disconnected iff $\widehat{G}$ is a torsion group (i.e., all elements of $\widehat{G}$ are of finite order).
(*Hint to* **(b)**: if G is totally disconnected, then $\operatorname{Ker}\chi$ is clopen in G for every $\chi \in \widehat{G}$.)

9.9. Let G be a compact abelian group equipped with the normalized Haar measure. Show that $\widehat{G}$ is a total subset of $C(G)$ and an orthonormal basis of $L^2(G)$.

9.10. Let G be a locally compact abelian group such that the Fourier transform $\mathscr{F} \colon L^1(G) \to C_0(\widehat{G})$ is surjective. Prove that G is finite. (*Hint:* look at $\mathscr{F}^*$.)

9.11*. Show that the characteristic function of $\mathbb{N} \subset \mathbb{Z}$ does not belong to the Fourier-Stieltjes algebra $B(\mathbb{Z})$. Thus the Fourier transform $\mathscr{F} \colon M(\mathbb{T}) \to C_b(\mathbb{Z})$ is not surjective.

9.12. Let G be a locally compact group, and let $f \in L^1(G)$, $g \in L^2(G)$. Show that
(a) the convolution $f * g$ is defined a.e. on G, belongs to $L^2(G)$, and $\|f * g\|_2 \leqslant \|f\|_1 \|g\|_2$;
(b) If G is abelian, then $(f * g)\widehat{\ } = \hat{f}\hat{g}$.

9.13. Let G be a locally compact abelian group. Show that
(a) if $f, g \in L^2(G)$, then $(fg)\widehat{\ } = \hat{f} * \hat{g}$;
(b) $A(G) = L^2(G) * L^2(G)$ (where $A(G)$ is the Fourier algebra of G).

9.14. Let G be a locally compact abelian group, and let $B^1(G) = B(G) \cap L^1(G)$. Show that $B^1(G)$ is an algebra under convolution and under pointwise multiplication.

Definition 9.1. Let G be a locally compact group. The *(full) C^*-algebra* of G, $C^*(G)$, is defined to be the C^*-envelope of $L^1(G)$ (see Definition 8.1). The *reduced C^*-algebra* of G is the C^*-subalgebra $C^*_r(G)$ of $\mathscr{B}(L^2(G))$ generated by the image of the left regular representation $\lambda\colon L^1(G) \to \mathscr{B}(L^2(G))$, $\lambda(f)g = f * g$ (see Exercise 9.12 (a)).

The universal property of $C^*(G)$ yields a canonical $*$-homomorphism $C^*(G) \to C^*_r(G)$.

9.15. Prove that, if G is abelian, then the canonical map $C^*(G) \to C^*_r(G)$ is an isomorphism.

9.16. Define covariant functors $G \mapsto M(G)$ and $G \mapsto C_b(\widehat{G})$ from the category of locally compact abelian groups to the category of unital Banach $*$-algebras. Show that the Fourier transform is a natural transformation between them.

9.17. Let G be a locally compact abelian group. Given a closed subgroup $H \subset G$, define the *annihilator* of H by $H^\perp = \{\chi \in \widehat{G} : \chi|_H = 1\}$.
(a) Show that $H^{\perp\perp} = H$ (here $\widehat{\widehat{G}}$ is canonically identified with G).
(b) Construct topological isomorphisms $\widehat{G/H} \cong H^\perp$, $\widehat{H} \cong \widehat{G}/H^\perp$.
(c) A short exact sequence $1 \to G' \xrightarrow{i} G \xrightarrow{p} G'' \to 1$ of locally compact abelian groups is *strictly exact* if i is a topological embedding and p is a quotient map. Show that a sequence of the above form is strictly exact iff the dual sequence is strictly exact.