Positive definite functions and positive functionals

(exercises for Lectures 14–15)

Convention. Throughout, all locally compact groups are assumed to be second countable.

8.1. Let G be a finite abelian group. Recall (see Lecture 1) that the dual group $\widehat{G}$ is a vector space basis of the space $\mathrm{Fun}(G)$ of all functions on G. Without using the Fourier transform, show that a function $\varphi = \sum_{\chi\in\widehat{G}} c_\chi \chi$ is positive definite iff $c_\chi \geqslant 0$ for all χ.

8.2. Let G be a locally compact group, and let $\mathscr{P}(G)$ be the set of all continuous, positive definite functions on G. Show that
(a) If $\varphi \in \mathscr{P}(G)$ and $\chi\colon G \to \mathbb{T}$ is a continuous character, then $\chi\varphi \in \mathscr{P}(G)$;
(b) If G is abelian and $\varphi, \psi \in \mathscr{P}(G)$, then $\varphi\psi \in \mathscr{P}(G)$.

8.3. Let G be a group, and let π be a unitary representation of G on a Hilbert space H. Show that, for every $v \in H$, the function $\varphi(x) = \langle \pi(x)v \,|\, v\rangle$ is positive definite.

8.4. Let G be a locally compact group, and let $f, g \in L^2(G)$. Show that
(a) the convolution $f * Sg$ (where $(Sg)(x) = g(x^{-1})$) is defined everywhere on G, belongs to $C_0(G)$, and satisfies $\|f * Sg\|_\infty \leqslant \|f\|_2 \|g\|_2$;
(b) $f * \overline{Sf}$ is positive definite.

8.5. Let A be a commutative Banach $*$-algebra, $\mathrm{Max}_h(A) \subset \mathrm{Max}(A)$ be the set of all maximal modular $*$-ideals of A, and $\hat{A}_h \subset \hat{A}$ be the set of all nonzero $*$-characters of A. Show that
(a) the map $\hat{A}_h \to \mathrm{Max}_h(A)$, $\chi \mapsto \mathrm{Ker}\,\chi$, is a bijection;
(b) $\hat{A}_h$ is closed in $\hat{A}$.

8.6 (*another generalization of Bochner's theorem)***.** Let A be a commutative Banach $*$-algebra with a bounded approximate identity. Denote by $\Gamma_h\colon A \to C_0(\hat{A}_h)$ the composition of the Gelfand transform $\Gamma\colon A \to C_0(\hat{A})$ with the restriction map $C_0(\hat{A}) \to C_0(\hat{A}_h)$. Show that the dual map Γ_h^* maps the set $M(\hat{A}_h)_{\mathrm{pos}}$ of finite positive Radon measures on $\hat{A}_h$ bijectively onto the set A^*_{pos} of positive functionals on A. (*Hint:* for a hermitian A, see the lectures.)

8.7 (*GNS construction*[1] *for untopologized groups)***.** Let G be a group, let φ be a positive definite function on G, and let α_φ denote the respective positive linear functional on the group algebra $\mathbb{C}G$ given by $\alpha_\varphi(\delta_x) = \varphi(x)$ $(x \in G)$. Define a semi-inner product on $\mathbb{C}G$ by $\langle f \,|\, g\rangle_\varphi = \alpha_\varphi(g^* f)$. Let $N_\varphi = \{f \in \mathbb{C}G : \langle f \,|\, f\rangle_\varphi = 0\}$, and let H_φ denote the Hilbert space completion of $\mathbb{C}G/N_\varphi$ w.r.t. the inner product induced by $\langle - \,|\, -\rangle_\varphi$.
(a) Show that there exists a unitary representation π_φ of G on H_φ uniquely determined by $\pi_\varphi(x)(\delta_y + N_\varphi) = \delta_{xy} + N_\varphi$ $(x, y \in G)$.
(b) Show that there exists $v \in H_\varphi$ such that $\varphi(x) = \langle \pi_\varphi(x)v \,|\, v\rangle$ for all $x \in G$. (Thus every positive definite function on G has the form described in Exercise 8.3.)

8.8 (*GNS construction for unital Banach $*$-algebras)***.** Let A be a unital Banach $*$-algebra, and let ω be a positive linear functional on A.
(a) Show that for every selfadjoint element $a \in A$ with $\|a\| < 1$ there exists a selfadjoint element $b \in A$ such that $b^2 = 1 - a$. (*Hint:* use the Taylor series for $\sqrt{1-z}$.)
(b) Show that ω is continuous and satisfies $\|\omega\| = \omega(1)$. (*Hint*: use **(a)**.)
(c) Show that for all $a, b \in A$ we have $|\omega(b^*ab)| \leqslant \|a\|\omega(b^*b)$. (*Hint*: use **(b)**.)

[1]GNS is for Gelfand, Naimark, and Segal.

(d) Define a semi-inner product on A by $\langle a \mid b\rangle_\omega = \omega(b^*a)$. Let $N_\omega = \{a \in A : \langle a \mid a\rangle_\omega = 0\}$, and let H_ω denote the Hilbert space completion of A/N_ω w.r.t. the inner product induced by $\langle - \mid - \rangle_\omega$. Show that there exists a $*$-representation π_ω of A on H_ω uniquely determined by $\pi_\omega(a)(b+N_\omega) = ab + N_\omega$ $(a, b \in A)$. (*Hint*: use **(c)**.)
(e) Show that there exists $v \in H_\omega$ such that $\omega(a) = \langle \pi_\omega(a)v \mid v\rangle$ for all $a \in A$.

8.9 (*GNS construction for nonunital Banach $*$-algebras)*. Let A be a unital Banach $*$-algebra, and let ω be a positive linear functional on A.
(a) Show that Part **(d)** of Exercise 8.8 holds even if A is not assumed to be unital. (*Hint:* for every $b \in A$, the map $a \mapsto \omega(b^*ab)$ is a positive functional on A_+.)

From now on, we assume that A has a bounded approximate identity and that ω is continuous[1].

(b) Show that $\omega(a^*) = \overline{\omega(a)}$ and that there exists $C > 0$ such that $|\omega(a)|^2 \leqslant C\|a^*a\|$ $(a \in A)$.
(c) Extend ω to a linear functional ω_+ on A_+ by letting $\omega_+(1_+) = C$, where C is as in **(b)**. Show that ω_+ is positive.
(d) Show that the inclusion of A into A_+ induces an isometric isomorphism between H_ω and H_{ω_+}. Thus the restriction of π_{ω_+} to A can be identified with π_ω.
(e) Show that Part **(e)** of Exercise 8.8 holds even if A is not assumed to be unital.

8.10 (*GNS construction for locally compact groups)*. Let G be a locally compact group, let φ be a function of positive type on G, and let α_φ denote the respective positive linear functional on $L^1(G)$ given by $\alpha_\varphi(f) = \int_G f\varphi\, d\mu$ (where μ is a fixed Haar measure). Let $H_\varphi = H_{\alpha_\varphi}$, and let $\pi_\varphi = \pi_{\alpha_\varphi}$ denote the respective GNS representation of $L^1(G)$ on H_φ (see Exercise 8.9).
(a) Show that there exists a unitary representation π_φ of G on H_φ uniquely determined by $\pi_\varphi(x)(f + N_\varphi) = L_x f + N_\varphi$ $(x \in G,\ f \in L^1(G))$. Moreover, π_φ is continuous in the sense that, for every $h \in H_\varphi$, the map $G \to H_\varphi$, $x \mapsto \pi_\varphi(x)h$, is continuous.
(b) Show that there exists $v \in H_\varphi$ such that $\varphi(x) = \langle \pi_\varphi(x)v \mid v\rangle$ for almost all $x \in G$. (In particular, every function of positive type on G is a.e. equal to a unique continuous, positive definite function.)

Definition 8.1. Let A be a Banach $*$-algebra. A *C^*-envelope* of A is a pair $(C^*(A), \theta_A)$ consisting of a C^*-algebra $C^*(A)$ and a $*$-homomorphism $\theta_A \colon A \to C^*(A)$ such that for each C^*-algebra B and each $*$-homomorphism $\varphi\colon A \to B$ there exists a unique $*$-homomorphism $\psi\colon C^*(A) \to B$ satisfying $\psi \circ \theta_A = \varphi$.

Clearly, if $C^*(A)$ exists, then it is unique up to a unique (necessarily isometric) $*$-isomorphism over A. The next exercise shows that $C^*(A)$ always exists.

8.11. Let A be a Banach $*$-algebra. Show that there exists a largest C^*-seminorm $\|\cdot\|_*$ on A, and that the completion of $A/\{a : \|a\|_* = 0\}$ w.r.t. the respective quotient norm is a C^*-envelope of A.

8.12. Let A be a commutative Banach $*$-algebra. Show that $(C_0(\hat{A}_h), \Gamma_h)$ (see Exercise 8.6) is a C^*-envelope of A.

8.13. Find the C^*-envelope of **(a)** $C^n[a,b]$; **(b)** $\mathscr{A}(\bar{\mathbb{D}})$; **(c)** $C_0(X)_\sigma$ (see Exercise 7.8).

8.14 (*yet another generalization of Bochner's theorem)*. Let A be a Banach $*$-algebra with a bounded approximate identity. Show that θ_A^* maps $C^*(A)^*_{\text{pos}}$ bijectively onto A^*_{pos}.

(*Hint:* the problem of extending a positive functional on A to a positive functional on $C^*(A)$ reduces to the universal property of $C^*(A)$ via the GNS construction.)

Note that Exercise 8.14 implies Exercise 8.6 modulo Exercise 8.12.

[1] In fact, a positive functional on a Banach $*$-algebra with a bounded approximate identity is automatically continuous. The proof is based on a nontrivial factorization theorem due to Cohen.