C^*-algebras and hermitian $*$-algebras

(EXERCISES FOR LECTURES 11-12)

7.1. Let X be a locally compact Hausdorff topological space, and let X_+ denote the one-point compactification of X. For each $f \in C_0(X)$, define $f_+ \colon X_+ \to \mathbb{C}$ by $f_+(x) = f(x)$ for $x \in X$ and $f_+(\infty) = 0$. Prove that f_+ is continuous, and that the map $C_0(X)_+ \to C(X_+)$, $f + \lambda 1_+ \mapsto f_+ + \lambda$, is an isometric $*$-isomorphism. (Here we assume that $C_0(X)_+$ is equipped with the canonical C^*-norm extending the supremum norm on $C_0(X)$.)

7.2. Let A denote any of the following algebras: **(a)** $C^1[a,b]$; **(b)** $\mathscr{A}(\bar{\mathbb{D}})$; **(c)** $\ell^1(\mathbb{Z})$; **(d)** $L^1(\mathbb{T})$; **(e)** $L^1(\mathbb{R})$. Does there exist a norm and an involution on A making it into a C^*-algebra? (We do not assume that the new norm is equivalent to the original norm.)

7.3. Let A and B be C^*-algebras. Show that if B is commutative, then each homomorphism from A to B is a $*$-homomorphism. Does the above result hold without the commutativity assumption?

7.4. Let $A = C^1[0,1]$. **(a)** Is A hermitian? **(b)** Does the identity $\|a\| = r(a)$ hold in A?

7.5. Let $A = \mathscr{A}(\bar{\mathbb{D}})$. **(a)** Is A hermitian? **(b)** Does the identity $\|a\| = r(a)$ hold in A?

7.6. Let $A = \ell^1(\mathbb{Z})$. **(a)** Is A hermitian? **(b)** Does the identity $\|a\| = r(a)$ hold in A?

7.7. Let $A = \mathscr{A}^+(\bar{\mathbb{D}})$ consist of those $f \in \mathscr{A}(\bar{\mathbb{D}})$ for which $\sum_n |c_n(f)| < \infty$, where $c_n(f)$ is the nth Taylor coefficient of f at 0.
(a) Show that A is a $*$-subalgebra of $\mathscr{A}(\bar{\mathbb{D}})$.
(b)* Prove that $A \neq \mathscr{A}(\bar{\mathbb{D}})$.
(c) Show that A is a Banach $*$-algebra w.r.t. the norm $\|f\| = \sum_{n \geqslant 0} |c_n(f)|$.
(d) Is A hermitian? **(e)** Does the identity $\|a\| = r(a)$ hold in A?

7.8. Let X be a locally compact Hausdorff topological space, and let $\sigma \colon X \to X$ be a continuous map such that $\sigma^2 = \mathbf{1}$. Define an involution on $C_0(X)$ by $f^*(x) = \overline{f(\sigma(x))}$ $(x \in X)$. The resulting Banach $*$-algebra will be denoted by $C_0(X)_\sigma$.
(a) Describe all $*$-characters of $C_0(X)_\sigma$.
(b) When is $C_0(X)_\sigma$ hermitian?

7.9 (*Beurling algebras*)**.** Let $\omega = (\omega_n)_{n \in \mathbb{Z}}$ be a sequence of positive numbers such that $\omega_0 = 1$ and $\omega_{m+n} \leqslant \omega_m \omega_n$ for all $m, n \in \mathbb{Z}$. (Such an ω is called a *weight* on $\mathbb{Z}$.)
(a) Show that the limit $\lim\limits_{n \to +\infty} \omega_n^{1/n}$ exists and is equal to $\inf\limits_{n \geqslant 1} \omega_n^{1/n}$.
(b) Let

$$\ell^1(\mathbb{Z}, \omega) = \Big\{ f \colon \mathbb{Z} \to \mathbb{C} : \|f\| = \sum_{n \in \mathbb{Z}} |f(n)| \omega_n < \infty \Big\}.$$

Show that $\ell^1(\mathbb{Z}, \omega)$ is a commutative unital Banach algebra under convolution.
(c) Show that the maximal spectrum of $\ell^1(\mathbb{Z}, \omega)$ is homeomorphic to the annulus $K_{r,R} = \{z \in \mathbb{C} : r < |z| < R\}$ via the map $\chi \mapsto \chi(\delta_1)$, where δ_1 is the function on $\mathbb{Z}$ that is 1 at 1, 0 elsewhere. Give explicit formulas for r and R in terms of ω.
(d) Give a condition on ω that is necessary and sufficient for $\ell^1(\mathbb{Z}, \omega)$ to be a Banach $*$-algebra w.r.t. the involution $f^*(n) = \overline{f(-n)}$.
(e) Under Condition (d), characterize those $x \in K_{r,R}$ which correspond to $*$-characters.
(f) When is $\ell^1(\mathbb{Z}, \omega)$ hermitian?

7.10. Let A be a unital Banach $*$-algebra, and let $B \subset A$ be a closed $*$-subalgebra containing the identity of A.
(a) Show that, if B is hermitian, then it is spectrally invariant in A.
(b) Does (a) hold if we drop the assumption that B is hermitian?
(c) Show that, if B is hermitian, then the induced map $\operatorname{Max} A \to \operatorname{Max} B$ is onto.
(d) Does (c) hold if we drop the assumption that B is hermitian?

7.11. Let $\varphi\colon A \to B$ be a surjective $*$-homomorphism of C^*-algebras. Is it true that
(a) for each selfadjoint $b \in B$ there exists a selfadjoint $a \in A$ with $\varphi(a) = b$?
(b) for each unitary $b \in B$ there exists a unitary $a \in A$ with $\varphi(a) = b$?

7.12. Let H be a Hilbert space. Show that
(a) $T \in \mathscr{B}(H)$ is selfadjoint $\iff \langle Tx \,|\, x\rangle \in \mathbb{R}$ for all $x \in H$;
(b) $P \in \mathscr{B}(H)$ is an orthogonal projection onto a closed subspace of $H \iff P = P^* = P^2$;
(c) $T \in \mathscr{B}(H)$ is an isometry (i.e., $\|Tx\| = \|x\|$ for all $x \in H$) $\iff \langle Tx \,|\, Ty\rangle = \langle x \,|\, y\rangle$ for all $x, y \in H \iff T^*T = 1$;
(d) $U \in \mathscr{B}(H)$ is a bijective isometry $\iff$ U is a unitary element of $\mathscr{B}(H)$ (i.e., $U^*U = UU^* = 1$);
(e) give a geometric interpretation of the property $TT^* = 1$ for $T \in \mathscr{B}(H)$.

Definition 7.1. Let A be a C^*-algebra. An element $p \in A$ is a *projection* if $p = p^* = p^2$ (cf. Exercise 7.12 (b)).

7.13. Let A be a C^*-algebra, and let $u \in A$.
(a) Show that u^*u is a projection iff $uu^*u = u$. An element with the above property is called a *partial isometry*.
(b) Let H be a Hilbert space. Show that $u \in \mathscr{B}(H)$ is a partial isometry iff the restriction of u to $(\operatorname{Ker} u)^\perp$ is an isometry.