The group Banach algebras $L^1(G)$ and $M(G)$. Approximate identities

(EXERCISES FOR LECTURES 8–9)

Convention. Throughout, all locally compact groups are assumed to be second countable.

5.1. Show that **(a)** $C^n[a,b]$ ($n \geqslant 1$) and **(b)** $\mathscr{A}(\bar{\mathbb{D}})$ are Banach $*$-algebras, but are not C^*-algebras. (Recall that the involution on $C^n[a,b]$ is given by $f^*(t) = \overline{f(t)}$, while the involution on $\mathscr{A}(\bar{\mathbb{D}})$ is given by $f^*(z) = \overline{f(\bar{z})}$.)

5.2. Let G be a locally compact group. As was shown in the lectures, $L^1(G)$ is a Banach algebra under convolution.
(a) Show that $L^1(G)$ is a Banach $*$-algebra w.r.t. the involution $f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1})$ ($f \in L^1(G)$, $x \in G$).
(b) Show that $L^1(G)$ (equipped with the standard L^1-norm and with the involution defined in (a)) is not a C^*-algebra unless $G = \{e\}$.
(c) Show that $L^1(G)$ is unital if and only if G is discrete.

5.3. Let G be a locally compact group.
(a) Prove the associativity of the convolution on $M(G)$.
(b) Show that $M(G)$ is a Banach $*$-algebra w.r.t. the involution $\langle \nu^*, f\rangle = \overline{\langle \nu, \overline{Sf}\rangle}$ ($\nu \in M(G)$, $f \in C_0(G)$).
(c) Show that, for each $\nu \in M(G)$ and each Borel set $B \subset G$, we have $\nu^*(B) = \overline{\nu(B^{-1})}$.

5.4. Let G be a locally compact group, and let μ be a left Haar measure on G. Given $f \in L^1(G)$, define $f \cdot \mu \in M(G)$ by $\langle f \cdot \mu, g\rangle = \langle \mu, fg\rangle$ ($g \in C_0(G)$).
(a) Show that for each Borel set $B \subset G$ we have $(f \cdot \mu)(B) = \int_B f\, d\mu$.
(b) Show that the map $i\colon L^1(G) \to M(G)$, $f \mapsto f \cdot \mu$, is an isometric $*$-algebra homomorphism.
(c) Identify $L^1(G)$ with its canonical image in $M(G)$ (see (b) above). Show that $L^1(G)$ is a closed 2-sided ideal of $M(G)$, and that for each $f \in L^1(G)$, $\nu \in M(G)$, and for almost all $x \in G$ we have

$$(\nu * f)(x) = \int_G f(y^{-1}x)\, d\nu(y), \qquad (f * \nu)(x) = \int_G f(xy^{-1})\Delta(y^{-1})\, d\nu(y). \tag{1}$$

5.5. Given a locally compact group G and $x \in G$, let δ_x denote the Dirac measure on G concentrated at x. Show that, for each $\nu \in M(G)$ and each $x \in G$, we have

$$L_x\nu = \delta_x * \nu, \quad R_x\nu = \nu * \delta_{x^{-1}}, \quad \delta_x * \delta_y = \delta_{xy}.$$

In particular, δ_e is the identity of $M(G)$.

5.6. Show that $L^1(G)$ is commutative $\iff$ $M(G)$ is commutative $\iff$ G is commutative.

5.7. Let G be a locally compact group. Prove that
(a) the map $\alpha\colon \mathbb{C}G \to M(G)$, $x \in G \mapsto \delta_x \in M(G)$, is an injective algebra homomorphism;
(b) α is onto iff G is finite;
(c) α uniquely extends to an isometric homomorphism $\beta\colon \ell^1(G) \to M(G)$;
(d) β is onto iff G is discrete.

5.8. Let G be a nondiscrete locally compact group, and let μ be a left Haar measure on G.
(a) Prove that there are Banach space direct sum decompositions

$$M(G) = M_d(G) \oplus M_c(G), \quad M_c(G) = M_a(G) \oplus M_{cs}(G),$$

where

$$M_d(G) = \{\nu \in M(G) : |\nu|(G \setminus S) = 0 \text{ for an at most countable } S \subset G\}$$

is the subspace of *discrete* measures,

$$M_c(G) = \{\nu \in M(G) : \nu(\{x\}) = 0 \text{ for all } x \in G\}$$

is the subspace of *continuous* measures,

$$M_a(G) = \{\nu \in M(G) : \nu \ll \mu\}$$

is the subspace of measures that are *absolutely continuous* w.r.t. μ, and

$$M_{cs}(G) = \{\nu \in M_c(G) : \nu \perp \mu\}$$

is the subspace of *continuous singular* measures.
(b) Show that $M_d(G)$ is a closed $*$-subalgebra of $M(G)$ isomorphic to $\ell^1(G)$ via β (see Exercise 5.7), $M_c(G)$ is a closed two-sided $*$-ideal of $M(G)$, and $M_a(G)$ is a closed two-sided $*$-ideal of $M(G)$ isomorphic to $L^1(G)$ via i (see Exercise 5.4).

(*Hint to* **(a)**: use the Lebesgue-Radon-Nikodym theorem.)

5.9. Let A be a normed algebra, and let (e_α) be a bounded approximate identity in A. Show that
(a) if B is a normed algebra and $\varphi \colon A \to B$ is a continuous homomorphism such that $\overline{\varphi(A)} = B$, then $(\varphi(e_\alpha))$ is a bounded approximate identity in B;
(b) if A is a normed $*$-algebra, then $(e_\alpha^* e_\alpha)$ is a bounded approximate identity in A.

5.10. Let X be a locally compact Hausdorff topological space.
(a) Construct an approximate identity in $C_0(X)$.
(b) Show that $C_0(X)$ has a sequential approximate identity if and only if X is σ-compact.

5.11. Let H be a Hilbert space.
(a) Construct an approximate identity in $\mathscr{K}(H)$.
(b) Show that $\mathscr{K}(H)$ has a sequential approximate identity if and only if H is separable.

5.12. Show that the Banach algebra ℓ^1 equipped with the pointwise multiplication does not have a bounded approximate identity.

5.13. Show that the Banach algebra $A = \{f \in C^1[0,1] : f(0) = 0\}$ does not have an approximate identity.

5.14. Let G be a locally compact group, and let (u_α) be a Dirac net in $L^1(G)$. Show that (u_α) converges to $\delta_e \in M(G)$ w.r.t. the weak* topology on $M(G)$.

5.15. Let A be a normed algebra with a bounded approximate identity. Suppose that (e_n) is a sequential approximate identity in A.
(a) Is (e_n) necessarily bounded?
(b) Is (e_n) necessarily bounded under the assumption that A is a Banach algebra?