Locally compact spaces and groups. The Haar measure

(exercises for Lectures 4–7)

4.1. Let $(X_i)_{i\in I}$ be a family of topological spaces. Show that $\prod_{i\in I} X_i$ is locally compact if and only if each X_i is locally compact and only finitely many of the X_i's are noncompact.

4.2. Show that the counting measure on a nondiscrete Hausdorff topological space is not a Radon measure.

4.3. Let X be a locally compact, σ-compact Hausdorff topological space. Show that each outer Radon measure on X is inner regular on all Borel sets.

4.4. Let X be a locally compact, second countable Hausdorff topological space. Show that each Borel measure on X that is finite on compact sets is a Radon measure.

4.5. Let X be a locally compact topological space, and let $C_c(X)$ denote the space of all continuous compactly supported functions on X. Given a compact set $K \subset X$, let $C_K(X) = \{f \in C_c(X) : \operatorname{supp} f \subset K\}$. We endow $C_K(X)$ with the topology generated by the sup-norm $\|f\| = \sup_{x\in K} |f(x)|$. Show that every positive linear functional $I\colon C_c(X) \to \mathbb{C}$ is continuous on $C_K(X)$, for each compact set $K \subset X$. (Equivalently, this means that I is continuous w.r.t. the inductive limit topology on $C_c(X) = \varinjlim_K C_K(X)$.) Of course, you are not allowed to use the Riesz-Markov-Kakutani theorem, otherwise the exercise would be trivial...

4.6. Give an example of a locally compact group G and a left uniformly continuous function $f\colon G \to \mathbb{C}$ that is not right uniformly continuous.

4.7. Find explicitly the left and the right Haar measures on $\mathrm{GL}_n(\mathbb{R})$. (You can find an answer in many books, and you can easily check that it works, but I recommend you to deduce the formula for the Haar measure by using the method that was discussed at the lecture in the context of arbitrary Lie groups.)

4.8 (*the "$ax + b$" group*)**.** Let G be the group of all matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$, where $a \in \mathbb{R}^\times$ and $b \in \mathbb{R}$ (this group is isomorphic to the group of all affine transformations $x \mapsto ax + b$ of $\mathbb{R}$). Find explicitly the left and the right Haar measures on G. (You can find an answer in many books, and you can easily check that it works, but I recommend you to deduce the formula for the Haar measure by using the method that was discussed at the lecture in the context of arbitrary Lie groups.)

4.9 (*the Heisenberg group*)**.** Let G be the group of all matrices of the form $\left(\begin{smallmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix}\right)$, where $a, b, c \in \mathbb{R}$. Find explicitly the left and the right Haar measures on G. (You can find an answer in many books, and you can easily check that it works, but I recommend you to deduce the formula for the Haar measure by using the method that was discussed at the lecture in the context of arbitrary Lie groups.)

4.10. Let G be a locally compact group, and let $\chi\colon G \to \mathbb{R}_{>0}$ be a continuous homomorphism. Show that there exists a unique (up to a positive constant) positive Radon measure on G such that for each Borel set $B \subset G$ we have $\mu(xB) = \chi(x)\mu(B)$. (*Hint:* express μ in terms of a Haar measure on G.)

4.11. Let $p \in \mathbb{N}$ be a prime number. Show that the following definitions of the field $\mathbb{Q}_p$ of p-adic numbers and of the ring $\mathbb{Z}_p$ of p-adic integers are equivalent:

(i) $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$, $\mathbb{Q}_p$ is the field of fractions of $\mathbb{Z}_p$.

(ii) $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ w.r.t. the p-adic norm $|\cdot|_p$ given by $|x|_p = p^{-r}$, where $x = p^r a/b \in \mathbb{Q} \setminus \{0\}$, $a \in \mathbb{Z}$, $b \in \mathbb{N}$, $p \nmid a$, $p \nmid b$; $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leqslant 1\}$.
(iii) $\mathbb{Q}_p$ consists of all formal expressions of the form $x = \sum_{k=n}^{\infty} a_k p^k$, where $n \in \mathbb{Z}$ and $a_k \in \{0, 1, \ldots, p-1\}$; $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : a_k = 0 \ \forall k < 0\}$. (Define algebraic operations on such formal expressions! What is $|x|_p$ if x has the above form?)

Show that the projective limit topology on $\mathbb{Z}_p$ defined in (i) agrees with the norm topology defined in (ii). Prove that $\mathbb{Z}_p$ is compact and that $\mathbb{Q}_p$ is locally compact.

4.12. Let μ denote the Haar measure on $\mathbb{Q}_p$ normalized in such a way that $\mu(\mathbb{Z}_p) = 1$. Show that
(a) $\mu(\mathbb{B}_{p^k}(x)) = p^k$, where $\mathbb{B}_{p^k}(x)$ is the closed ball of radius p^k ($k \in \mathbb{Z}$) centered at $x \in \mathbb{Q}_p$.
(b) For each Borel set $B \subset \mathbb{Q}_p$ we have

$$\mu(B) = \inf\left\{ \sum_{i=1}^{\infty} p^{k_i} : B \subset \bigcup_{i=1}^{\infty} \mathbb{B}_{p^{k_i}}(x_i) \right\}.$$

4.13. Let $G = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ be the product of countably many copies of $\mathbb{Z}/2\mathbb{Z}$, and let μ denote the normalized Haar measure on G.
(a) Calculate $\mu(U)$ for a basic open set $U = \prod_{i=1}^{\infty} U_i$, where $U_i = \mathbb{Z}/2\mathbb{Z}$ for all but finitely many i.
(b) Define $f\colon G \to [0,1]$ by $f(a_1, a_2, \ldots) = \sum_i a_i 2^{-i}$. Show that f is onto and that $f^{-1}(x)$ is one point unless x is a dyadic rational, in which case $f^{-1}(x)$ consists of two points. Prove that the image of μ under f is the Lebesgue measure on $[0,1]$.
(c) Define $h\colon G \to [0,1]$ by $h(a_1, a_2, \ldots) = \sum_i 2a_i 3^{-i}$. Show that h is a homeomorphism of G onto the Cantor set. Prove that the image of μ under h is the Lebesgue-Stieltjes measure on $[0,1]$ associated to the Cantor function.

4.14. Let G be a locally compact group, and let μ be a positive Radon measure on G.
(a) Given a continuous function $f\colon G \to \mathbb{R}_{\geqslant 0}$, define a Radon measure $f\cdot\mu$ on G by $\langle f\cdot\mu, g\rangle = \langle \mu, fg\rangle$ ($g \in C_c(G)$). Show that for each $x \in G$ we have $L_x(f\cdot\mu) = L_x f \cdot L_x\mu$, where $L_x f$ and $L_x\mu$ are the left translates of f and μ, respectively. Prove a similar formula for the right translates.
(b) Define a Radon measure $S\mu$ on G by $\langle S\mu, g\rangle = \langle \mu, Sg\rangle$ ($g \in C_c(G)$), where $(Sg)(x) = g(x^{-1})$ ($x \in G$). Show that for each continuous function $f\colon G \to \mathbb{R}_{\geqslant 0}$ we have $S(f\cdot\mu) = Sf \cdot S\mu$.

4.15. Let G be a real Lie group. Show that the modular character Δ of G is given by $\Delta(x) = |\det \mathrm{Ad}_{x^{-1}}|$, where Ad is the adjoint representation of G.

4.16. Calculate the modular character of
(a) the "$ax+b$" group (see Exercise 5.8);
(b) the group of upper triangular 2×2-matrices.

4.17. Show that $\mathrm{SL}_2(\mathbb{R})$ is unimodular. (*Hint:* you do not need an explicit formula for the Haar measure on $\mathrm{SL}_2(\mathbb{R})$.)