

Harmonic analysis on a finite abelian group

(EXERCISES FOR LECTURE 1)

Convention. All vector spaces are over \mathbb{C} .

- 1.1. Prove that every character of a finite group is unitary.
- 1.2. Prove that a finite group has an injective character iff it is cyclic.
- 1.3. Describe all characters of (a) S_n ; (b) D_n ; (c) Q_8 .
- 1.4. Prove that, for a finite group, the intersection of the kernels of all the characters is the commutator subgroup (that is, the subgroup generated by all possible commutators $xyx^{-1}y^{-1}$).
- 1.5. Let G be a finite group, and let $\text{Fun}(G) = \mathbb{C}^G$ be the space of all functions on G . Recall that the convolution on $\text{Fun}(G)$ is a bilinear map uniquely determined by $\delta_x * \delta_y = \delta_{xy}$ ($x, y \in G$), where δ_x is the function equal to 1 at $x \in G$ and 0 elsewhere. Prove that for all $f, g \in \text{Fun}(G)$ we have

$$(f * g)(x) = \sum_{y \in G} f(y)g(y^{-1}x) \quad (x \in G).$$

- 1.6. Let G be a finite abelian group of order n , and let $\pi: G \rightarrow \text{GL}(V)$ be a representation on a finite-dimensional vector space V . For every $\chi \in \widehat{G}$ define $V_\chi = \{v \in V : \pi(x)v = \chi(x)v \ \forall x \in G\}$. Define an operator P_χ on V by

$$P_\chi = \frac{1}{n} \sum_{x \in G} \overline{\chi(x)} \pi(x).$$

- (a) Prove that $P_\chi P_\tau = \delta_{\chi\tau} P_\chi$, $\sum_{\chi \in \widehat{G}} P_\chi = \mathbf{1}_V$, and $\text{Im } P_\chi = V_\chi$. Deduce that $V = \bigoplus_{\chi \in \widehat{G}} V_\chi$ and that P_χ is a projection onto V_χ along $\bigoplus_{\tau \neq \chi} V_\tau$.
- (b) Find V_χ in the case where π is the regular representation of G on $\text{Fun}(G)$ given by $(\pi(x)f)(y) = f(yx)$.

- 1.7. Let G be a finite abelian group. Prove that the following properties of a linear operator $T: \text{Fun}(G) \rightarrow \text{Fun}(G)$ are equivalent:

- (i) T is shift invariant (i.e., $T\pi(x) = \pi(x)T$ for all $x \in G$, where π is the regular representation from the previous exercise);
- (ii) there exists a function $f \in \text{Fun}(G)$ such that $Th = f * h$ for all $h \in \text{Fun}(G)$;
- (iii) all characters of G are eigenvectors for T .

- 1.8. Let G be a finite abelian group, and let H be a subgroup of G .

- (a) Construct an isomorphism $(G/H)^\wedge \cong H^\perp$, where $H^\perp = \{\chi \in \widehat{G} : \chi|_H = 1\}$ is the annihilator of H in \widehat{G} .

- (b) (*the Poisson summation formula*). Define $a: \text{Fun}(G) \rightarrow \text{Fun}(G/H)$ by $(af)(xH) = \sum_{y \in H} f(xy)$. Show that the diagram

$$\begin{array}{ccc} \text{Fun}(G) & \xrightarrow{\mathcal{F}_G} & \text{Fun}(G) \\ a \downarrow & & \downarrow \text{restr.} \\ \text{Fun}(G/H) & \xrightarrow{\mathcal{F}_H} & \text{Fun}(H^\perp) \end{array}$$

commutes. Deduce that for each $f \in \text{Fun}(G)$ we have

$$\sum_{y \in H} f(xy) = \frac{1}{(G:H)} \sum_{\chi \in H^\perp} \hat{f}(\chi) \overline{\chi(x)} \quad (x \in G).$$