Free compact quantum groups

(EXERCISES FOR LECTURE 16)

Definition 7.1. Let A be a unital C^*-algebra. An element $u \in M_n(A)$ is

(i) *biunitary* if u and $\bar{u}$ are unitary (where $(\bar{u})_{ij} = u_{ij}^*$ for all i, j);

(ii) *orthogonal* if u is invertible, $\bar{u} = u$, and $u^{-1} = u^\top$ (where $u^\top$ is the transpose of u);

(iii) *magic unitary* if each u_{ij} is an orthogonal projection, and $\sum_j u_{ij} = \sum_k u_{k\ell} = 1$ for all i, ℓ.

7.1. Let A be a C^*-algebra, and let $p_1, \ldots, p_n \in A$ be orthogonal projections such that $\sum_i p_i$ is an orthogonal projection. Show that the p_i's are pairwise orthogonal (i.e., $p_i p_j = 0$ for all $i \neq j$). Deduce that, if $u \in M_n(A)$ is a magic unitary, then the projections in each row and in each column of u are pairwise orthogonal.

7.2. Let A be a unital C^*-algebra.
(a) Show that each magic unitary in $M_n(A)$ is orthogonal, and that each orthogonal is biunitary.
(b) Show that, if A is commutative, then $a \mapsto \bar{a}$ is a $\mathbb{C}$-antilinear $*$-automorphism of $M_n(A)$. Deduce that each unitary in $M_n(A)$ is biunitary.
(c) Construct a unital C^*-algebra A and a unitary $u \in M_2(A)$ that is not a biunitary.

7.3. Show that
(a) $C(U_n) \cong C^*_{\mathrm{com}}\Big(u_{ij}, 1 \leqslant i, j \leqslant n \,\Big|\, u = (u_{ij}) \text{ is biunitary}\Big)$;
(b) $C(O_n) \cong C^*_{\mathrm{com}}\Big(u_{ij}, 1 \leqslant i, j \leqslant n \,\Big|\, u = (u_{ij}) \text{ is orthogonal}\Big)$;
(c) $C(S_n) \cong C^*_{\mathrm{com}}\Big(u_{ij}, 1 \leqslant i, j \leqslant n \,\Big|\, u = (u_{ij}) \text{ is a magic unitary}\Big)$.
Here $C^*_{\mathrm{com}}(\cdots)$ stands for the universal commutative unital C^*-algebra with prescribed generators and relations.

7.4. The *free unitary quantum group* $C^+(U_n)$ is defined by

$$C^+(U_n) = C^*\Big(u_{ij}, 1 \leqslant i, j \leqslant n \,\Big|\, u = (u_{ij}) \text{ is biunitary}\Big).$$

Show that there exists a unique comultiplication $\Delta\colon C^+(U_n) \to C^+(U_n) \otimes_* C^+(U_n)$ such that $(C^+(U_n), \Delta, u)$ is a compact matrix quantum group.

7.5. The *free orthogonal quantum group* $C^+(O_n)$ is defined by

$$C^+(O_n) = C^*\Big(u_{ij}, 1 \leqslant i, j \leqslant n \,\Big|\, u = (u_{ij}) \text{ is orthogonal}\Big).$$

Show that there exists a unique comultiplication $\Delta\colon C^+(O_n) \to C^+(O_n) \otimes_* C^+(O_n)$ such that $(C^+(O_n), \Delta, u)$ is a compact matrix quantum group.

7.6. The *free quantum permutation group* $C^+(S_n)$ is defined by

$$C^+(S_n) = C^*\Big(u_{ij}, 1 \leqslant i, j \leqslant n \,\Big|\, u = (u_{ij}) \text{ is a magic unitary}\Big).$$

Show that there exists a unique comultiplication $\Delta\colon C^+(S_n) \to C^+(S_n) \otimes_* C^+(S_n)$ such that $(C^+(S_n), \Delta, u)$ is a compact matrix quantum group.

Exercise 7.3 implies that we have canonical surjective morphisms of compact quantum groups $C^+(U_n) \to C(U_n)$, $C^+(O_n) \to C(O_n)$, and $C^+(S_n) \to C(S_n)$.

7.7. Let F_n denote the free group on n generators. Show that there exists a surjective morphism $C^+(U_n) \to C^*(F_n)$ of compact quantum groups. Deduce that $C^+(U_n)$ is noncommutative for $n \geqslant 2$.

7.8. Let L_n denote the free product of n copies of $\mathbb{Z}/2\mathbb{Z}$. Show that there exists a surjective morphism $C^+(O_n) \to C^*(L_n)$ of compact quantum groups. Deduce that $C^+(O_n)$ is noncommutative for $n \geqslant 2$.

7.9. Show that the canonical morphism $C^+(S_2) \to C(S_2)$ is an isomorphism.

7.10. Show that $C^+(S_n)$ is noncommutative and infinite-dimensional for $n \geqslant 4$. As a corollary, $C^+(S_n) \to C(S_n)$ is not an isomorphism.

Hint: for any pair p, q of orthogonal projections, the matrix

$$\begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

is a magic unitary.