C^*-envelopes

(EXERCISES FOR LECTURES 13–14)

5.1. Let $A = \mathbb{C}[t^{\pm 1}]$ be the Laurent polynomial algebra with involution $t^* = t^{-1}$. Prove that $C^*(A) \cong C^*(u \mid u \text{ is unitary}) \cong C(\mathbb{T})$.

5.2. Let $A = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ be the Laurent polynomial algebra with involution $t_i^* = t_i^{-1}$ ($i = 1, \ldots, n$). Prove that $C^*(A) \cong C^*_{\mathrm{com}}(u_1, \ldots, u_n \mid u_i \text{ are unitaries}) \cong C(\mathbb{T}^n)$.

5.3. Let $A = \mathbb{C}[t^{\pm 1}]$ be the Laurent polynomial algebra with involution $t^* = t$. Prove that $C^*(A)$ does not exist.

5.4. Let A be the universal unital $*$-algebra generated by selfadjoint elements p, q with relation $[p, q] = i1$ (the *Weyl algebra*). Show that $C^*(A) = 0$.

5.5. Let $\mathrm{SL}_2 = \mathrm{SL}_2(\mathbb{C})$, and let $\mathscr{O}(\mathrm{SL}_2)$ be the algebra of regular functions on SL_2. Define $a, b, c, d \in \mathscr{O}(\mathrm{SL}_2)$ by $a(g) = g_{11}$, $b(g) = g_{12}$, $c(g) = g_{21}$, $d(g) = g_{22}$. Define $\sigma \in \mathrm{Aut}(\mathrm{SL}_2)$ by $\sigma(g) = (\overline{g^\top})^{-1}$. Show that

(a) The rule $f^*(g) = \overline{f(\sigma(g))}$ defines an involution on $\mathscr{O}(\mathrm{SL}_2)$.

(b) The involution on $\mathscr{O}(\mathrm{SL}_2)$ is uniquely determined by $a^* = d$, $b^* = -c$.

(c) There exits bijections $\mathrm{SL}_2 \cong \mathrm{Hom}_{\mathsf{Alg}}(\mathscr{O}(\mathrm{SL}_2), \mathbb{C})$ and $\mathrm{SU}_2 \cong \mathrm{Hom}_{*\text{-}\mathsf{Alg}}(\mathscr{O}(\mathrm{SL}_2), \mathbb{C})$ (where Alg and $*$-Alg denote the category of unital algebras and of unital $*$-algebras, respectively).

(d) $C(\mathrm{SU}_2)$ is the C^*-envelope of $\mathscr{O}(\mathrm{SL}_2)$.

5.6 (the *rotation algebra* or the *quantum* 2-*torus*)**.** Given $\theta \in \mathbb{R}$, let $\mathscr{A}_\theta$ denote the universal unital $*$-algebra generated by unitaries u, v with relation $uv = e^{2\pi i\theta} vu$. Let also $A_\theta = C^*(\mathscr{A}_\theta)$.

(a) Show that $\mathscr{A}_0$ is $*$-isomorphic to the Laurent polynomial algebra $\mathbb{C}[u^{\pm 1}, v^{\pm 1}]$, and that A_0 is isometrically $*$-isomorphic to $C(\mathbb{T}^2)$.

(b) Show that for each $z = (\lambda, \mu) \in \mathbb{T}^2$ there exists a $*$-automorphism α_z of A_θ uniquely determined by $u \mapsto \lambda u$, $v \mapsto \mu v$. Prove that the resulting group homomorphism $\alpha \colon \mathbb{T}^2 \to \mathrm{Aut}(A_\theta)$ is continuous with respect to the strong operator topology on $\mathrm{Aut}(A_\theta)$.

(c) Define $E \colon A_\theta \to A_\theta$ by $E(a) = \int_{\mathbb{T}^2} \alpha_z(a)\, d\nu(z)$ (where ν is the normalized Haar measure on $\mathbb{T}^2$). Show that $\mathrm{Im}\, E = \mathbb{C}1$, that $E(1) = 1$, and that $E(u^k v^\ell) = 0$ unless $k = \ell = 0$.

(d) Show that the monomials $u^k v^\ell$ ($k, \ell \in \mathbb{Z}$) are linearly independent in A_θ. Deduce that the canonical map $\mathscr{A}_\theta \to A_\theta$ is injective.

(e) Show that if $E(a) = 0$ for some positive $a \in A_\theta$, then $a = 0$. (*Hint:* if $a \neq 0$, then take a state f on A such that $f(a) \neq 0$).

(f) Show that, if $\theta \notin \mathbb{Q}$, then α_z is inner whenever z belongs to a dense subset of $\mathbb{T}^2$.

(g) Show that, if $\theta \notin \mathbb{Q}$, then A_θ is simple (i.e., A_θ has no proper closed two-sided ideals other than 0).

(h) Show that, if $\theta \notin \mathbb{Q}$, then A_θ is isomorphic to the C^*-subalgebra of $\mathscr{B}(L^2(\mathbb{T}))$ generated by two operators U, V given by $(Uf)(z) = zf(z)$ and $(Vf)(z) = f(e^{-2\pi i\theta} z)$ ($f \in L^2(\mathbb{T})$, $z \in \mathbb{T}$).

(i) Does (h) hold if $\theta \in \mathbb{Q}$?

Recall that a bounded linear operator V on a Hilbert space is an isometry (that is, $\|Vx\| = \|x\|$ for all $x \in H$, or, equivalently, $\langle Vx \mid Vy \rangle = \langle x \mid y \rangle$ for all $x, y \in H$) iff $V^*V = \mathbf{1}$. If A is a unital $*$-algebra, then we say that $v \in A$ is an *isometry* if $v^*v = 1$.

Theorem 5.1 (Wold, von Neumann)**.** *Let V be an isometry on a Hilbert space. Then V is unitarily equivalent to $\dot{\bigoplus}_{i\in I} V_i$, where each V_i is either a unitary operator on a Hilbert space H_i or the right shift on $H_i = \ell^2$.*

Note that a Hilbert direct sum of unitary operators is unitary. So we may assume that at most one operator in the family $\{V_i\}$ is unitary.

5.7 (*Coburn's theorem*)**.** Let $A = C^*(u \mid u^*u = 1)$ be the universal C^*-algebra generated by an isometry. Prove that A is isomorphic to the Toeplitz algebra (see Exercise 3.8).

Hint: use the Wold–von Neumann theorem.