Tensor products

(EXERCISES FOR LECTURES 11–13)

4.1. Let H_1 and H_2 be infinite-dimensional inner product spaces. Prove that the algebraic tensor product $H_1 \otimes H_2$ is not complete w.r.t. the canonical inner product.

4.2. Let I be a set, and let H be a Hilbert space. Prove that there exists a unitary isomorphism $\ell^2(I) \dot{\otimes} H \to \ell^2(I, H) = \dot{\bigoplus}_{i \in I} H$ uniquely determined by $(c_i) \otimes h \mapsto (c_i h)$.

4.3. Let (X, μ) and (Y, ν) be σ-finite measure spaces. Prove that there exists a unitary isomorphism $L^2(X, \mu) \dot{\otimes} L^2(Y, \nu) \to L^2(X \times Y, \mu \times \nu)$ uniquely determined by $f \otimes g \mapsto ((x, y) \mapsto f(x)g(y))$.

4.4. Let A, B, C be C^*-algebras.
(a) Construct isometric $*$-isomorphisms $A \otimes_* B \cong B \otimes_* A$ and $A \otimes_* (B \otimes_* C) \cong (A \otimes_* B) \otimes_* C$.
(b) Do the same for $\otimes_{\max}$.

4.5. Let H_1 and H_2 be infinite-dimensional Hilbert spaces. Is the canonical embedding $\mathscr{B}(H_1) \otimes_* \mathscr{B}(H_2) \hookrightarrow \mathscr{B}(H_1 \dot{\otimes} H_2)$ an isomorphism?

4.6. Let H be an infinite-dimensional separable Hilbert space, and let (e_i) be an orthonormal basis of H. Show that there exists a unique bounded linear map $\theta \colon \mathscr{K}(H) \to \mathscr{K}(H)$ (the *transpose map*) such that $\langle \theta(T) e_j \,|\, e_i \rangle = \langle T e_i \,|\, e_j \rangle$ for all i, j and all $T \in \mathscr{K}(H)$. Prove that $\mathbf{1} \otimes \theta \colon \mathscr{K}(H) \otimes \mathscr{K}(H) \to \mathscr{K}(H) \otimes \mathscr{K}(H)$ is unbounded with respect to the spatial C^*-norm on $\mathscr{K}(H) \otimes \mathscr{K}(H)$.

4.7. Let H be an infinite-dimensional Hilbert space. Is the multiplication map $\mathscr{K}(H) \otimes \mathscr{K}(H) \to \mathscr{K}(H)$ bounded with respect to the spatial C^*-norm on $\mathscr{K}(H) \otimes \mathscr{K}(H)$?

4.8. Let X be a locally compact Hausdorff topological space, and let E be a Banach space. Prove that the map $C_0(X) \otimes E \to C_0(X, E)$, $f \otimes v \mapsto (x \mapsto f(x)v)$, has dense image.

Hint: partitions of unity.

4.9. Let X and Y be locally compact Hausdorff topological spaces. Construct an isometric $*$-isomorphism $C_0(X, C_0(Y)) \cong C_0(X \times Y)$.

4.10. Let A and B be $*$-algebras. A linear map $\varphi \colon A \to B$ is *positive* if $\varphi(A_{\text{pos}}) \subset B_{\text{pos}}$. Show that each positive linear map between C^*-algebras is continuous.

4.11. Let A and B be C^*-algebras.
(a) Let π be a nondegenerate $*$-representation of the algebraic tensor product $A \otimes B$ on a Hilbert space H. Show that there exists a unique pair (π_A, π_B) of nondegenerate $*$-representations of A and B on H such that for all $a \in A$, $b \in B$ we have $[\pi_A(a), \pi_B(b)] = 0$ and $\pi(a \otimes b) = \pi_A(a)\pi_B(b)$.
(b) Deduce that for each $u \in A \otimes B$ we have $\|u\|_{\max} < \infty$, and that $\|a \otimes b\|_{\max} = \|a\|\|b\|$ ($a \in A$, $b \in B$). (This was proved at the lecture under the assumption that A and B are unital.)

Hint to (a): define $\pi_A(a) = \text{SOT-}\lim \pi(a \otimes e_\lambda)$, where (e_λ) is an approximate identity in B. To show that the limit exists, use Exercise 4.10 to show that the map $b \mapsto \pi(a \otimes b)$ is bounded, and deduce that the net $\pi(a \otimes e_\lambda)$ is bounded.

4.12. Let A, B, C be C^*-algebras, and let $\pi \colon A \otimes B \to C$ be a $*$-homomorphism. Do there always exist $*$-homomorphisms $\varphi \colon A \to C$ and $\psi \colon B \to C$ such that for all $a \in A$, $b \in B$ we have $\pi(a \otimes b) = \varphi(a)\psi(b)$? (For $C = \mathscr{B}(H)$, the answer is yes by the previous exercise.)